

Mass, zero mass and ... nophysics

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Abstract

In this paper we demonstrate that massless particles cannot be considered as limiting case of massive particles. Instead, the usual symmetry structure based on semisimple groups like $U(1)$, $SU(2)$ and $SU(3)$ has to be replaced by less usual solvable groups like the minimal nonabelian group sol_2 . Starting from the proper orthochronous Lorentz group $\text{Lor}_{1,3}$ we extend Wigner's little group by an additional generator, obtaining the maximal solvable or Borel subgroup $\text{Bor}_{1,3}$ which is equivalent to the Kronecker sum of two copies of sol_2 , telling something about the helicity of particle and antiparticle states.

1 Introduction

In his paper “Sur la dynamique de l’électron” from July 1905, Henri Poincaré formulates the Principle of Relativity, introduces the concepts of Lorentz transformation and Lorentz group, and postulates the covariance of laws of Nature under Lorentz transformations [1]. In 1939, Eugène Wigner analysed the unitary representations of the inhomogeneous Lorentz group [2]. However, there are also nonunitary representations. As Wigner pointed out, the irreducible representations of the Lorentz group are important because they define the types of particles. The Lorentz symmetry is expressed by introducing the commutative diagram for the covariant wave functions ψ^C ,

$$\begin{array}{ccc} \psi^C : \mathbb{E}_{1,3} \ni p & \longrightarrow & \psi^C(p) \\ U(\Lambda) \downarrow & \downarrow \Lambda & \downarrow T(\Lambda) \\ U(\Lambda)\psi^C : \Lambda x \ni \Lambda p & \longrightarrow & T(\Lambda)\psi^C(p) \end{array} \quad (1)$$

implying $(U(\Lambda)\psi^C)(\Lambda p) = T(\Lambda)\psi^C(p)$. Here $\text{Lor}_{1,3} \ni \Lambda \rightarrow T(\Lambda)$ is the finite dimensional nonunitary representation.

Let the vectors $|p, \lambda\rangle$ with four-momentum p and independent parameter λ form a complete orthogonal basis of the irreducible representation of the Poincaré group. In this basis, $U(\Lambda)$ is always expressed by [3]

$$U(\Lambda) = Q(W(\Lambda, p)) P(\Lambda),$$

where Q is diagonal with respect to λ , P is diagonal with respect to p , and $W(\Lambda, p)$ is the Thomas–Wigner rotation.

The transformation law for Wigner’s wave function ψ^W reads

$$(U(\Lambda)\psi^W)(p, \lambda) = \sum_{\lambda'} Q_{\lambda\lambda'} (W(\Lambda, \Lambda^{-1}p)\psi^W(\Lambda^{-1}p, \lambda')). \quad (2)$$

Here $Q(W(\Lambda, p))$ is a representation of Wigner’s little group as a subrepresentation of some Lorentz group representation $T(\Lambda)$. The explicit relation between ψ^C and ψ^W is given in Refs. [4].

The great importance of Wigner's theory is that the classification of particles according to their Lorentz transformation properties is entirely determined by the representation of the little group as the subrepresentation of the representation of $\text{Lor}_{1,3} = \text{SO}_{1,3}^0$ [5].

The most important cases are

$$\begin{array}{c} \overset{\circ}{p} = (1, \underbrace{0, 0, 0}, \underbrace{0, 0, 0, 1}) \\ \text{Lg}(\overset{\circ}{p}) = \underbrace{\text{SO}_3}_{\text{Bor}_{1,3}} \underbrace{E(2)} \end{array}$$

2 The little group [2, 3, 4, 5]

The little group of the four-momentum $\overset{\circ}{p}$ or the stabiliser of $\overset{\circ}{p}$ is the maximal closed subgroup of $\text{Lor}_{1,3}$ defined as

$$\text{Lg } \overset{\circ}{p} = \{ \overset{\circ}{\Lambda} \in \text{Lor}_{1,3} : \overset{\circ}{\Lambda} \overset{\circ}{p} = \overset{\circ}{p} \}. \quad (3)$$

The orbit of $\overset{\circ}{p}$ is a subspace in $\mathbb{E}_{1,3}$,

$$\text{Orb } \overset{\circ}{p} = \{ \Lambda \overset{\circ}{p} : \Lambda \in \text{Lor}_{1,3} \} = \text{Lor}_{1,3} \overset{\circ}{p},$$

given as bijection $\text{Orb } \overset{\circ}{p} = \text{Lor } 1, 3 / \text{Lg } \overset{\circ}{p}$. For all $p \in \text{Orb } \overset{\circ}{p}$ the Thomas–Wigner rotation is given by

$$W(\Lambda, p) = L_{\Lambda p}^{-1} \Lambda L_{\Lambda p}, \quad (4)$$

where $L_p \in \text{Lor}_{1,3}$ is the representative of $p \in \text{Orb } \overset{\circ}{p}$.

The characteristic feature of the massive case is that the fixed vector $\overset{\circ}{p}$ can be chosen to be $\overset{\circ}{p} = (m, \vec{0})$, for which

$$\text{Lg } \overset{\circ}{p} = \text{SO}_3 \stackrel{\text{locally}}{=} \text{SU}_2$$

(SU_2 is the universal covering group of SO_3). Since SU_2 is compact and simply connected, all its finite-dimensional irreducible representations are single-valued, unitary and

parametrised by the eigenvalue s of the Casimir operator which can take on non-negative half-integer values

$$s = 0, \frac{1}{2}, 1, \frac{3}{2}, \dots$$

From the local equivalence, the equivalence of the Lie algebras can be derived, i.e.

$$\mathfrak{so}_3 = \mathfrak{ad} \mathfrak{su}_2 = \mathfrak{lg} \mathring{p}.$$

It is important that the little group SO_3 is the maximal compact simple subgroup of $\text{Lor}_{1,3}$ while \mathfrak{su}_2 (i.e. $\mathfrak{lg} \mathring{p}$) is the simplest semisimple Lie algebra, $\dim_{\mathbb{R}} \mathfrak{su}_2 = 3$.

For every irreducible unitary representation of the little group $\text{Lg} \mathring{p} = SO_3$ one can derive a corresponding induced representation of the Poincaré group, $\mathcal{P}_{1,3} = \mathcal{T}_{1,3} \rtimes \text{Lor}_{1,3}$. The irreducible representations of $\mathcal{P}_{1,3}$ are characterised by the pairs (m, s) , where the mass m is real and positive and the spin takes on the values $s = 0, \frac{1}{2}, 1, \dots$. The states within each irreducible representation are labelled by $\xi = -s, -s + 1, \dots, s$ which means that massive particles of spin s have $2s + 1$ degrees of freedom.

In the massless case $m = 0$ the representative vector \mathring{p} may be taken to be

$$\mathring{p} = (\omega_p, 0, 0, \omega_p), \quad \omega = |\vec{p}|. \quad (5)$$

To construct the little group one has to solve the defining equation

$$\mathring{\Lambda} \mathring{p} = \mathring{p}. \quad (6)$$

The result is the Euclidean group [2, 3, 4, 5]

$$\text{Lg} \mathring{p} = E(2) = \text{ISO}(2) = \mathcal{T}_2 \rtimes \text{SO}(2)$$

for which the double covering group is given by

$$\overline{E}(2) = \mathcal{T}_2 \rtimes U(1)$$

where \mathcal{T}_2 is the Abelian two-dimensional group of translations. Thus, the little group is solvable and non-compact, and the restrictions of the finite-dimensional representations of

$\text{Lor}_{1,3}$ to $E(2)$ are in general non-unitary. In fact, the only unitary, irreducible representation of $E(2)$ are one-dimensional, i.e. degenerate representations, since the subgroup \mathcal{T}_2 of translations has to be realised trivially [3],

$$E(2) \rightarrow E(2)/\mathcal{T}_2 = \text{SO}(2). \quad (7)$$

The requirement that the representations are at most double-valued implies that only the representations $\text{SO}(2) \rightarrow U^{(j)}(\text{SO}(2))$, $j = 0, \pm\frac{1}{2}, \pm 1, \dots$ are allowed. This one-dimensional (internal) freedom of massless particles is usually called the helicity. Since all the unitary representations on the orbits $p^2 = 0$ are induced by the non-faithful one-dimensional representation of the little group $E(2)$, massless particles are characterised by a discrete helicity

$$\lambda = 0, \pm\frac{1}{2}, \pm 1, \dots \quad (8)$$

Notice that if the parity is included, the helicity takes on two values, λ and $-\lambda$. For example, the two states $\lambda = \pm 1$ are then referred to as left-handed ($\lambda = -1$) and right-handed ($\lambda = +1$) photons.

2.1 The Borel subgroup [6]

Due to the unitarity of representations of the little group $\text{Lg } \overset{\circ}{p} = E(2)$, zero-mass particles have only a single value for the helicity (if the parity is not taken into account). Suppose, the most general determined, relativistically invariant, first order single particle equation is of the form

$$(\beta^\mu \partial_\mu + \rho) \psi(x) = 0. \quad (9)$$

Then there is a simple criterion by D. Kwoh under which this equation will have zero-mass solutions [7],

Kwoh's lemma: A necessary condition that Eq. (9) has a zero-mass solution is that

$$\det(-i\beta^\mu p_\mu + \lambda\rho) = 0 \quad (10)$$

for all real λ and all light-like p , i.e. all p such that $p^2 = 0$.

If Eq. (9) is a defining equation for a single massless particle, Kwoh's lemma states the gauge invariance $p \rightarrow \lambda p$ as a very special property of the theory. Therefore, it seems to be reasonable (at least mathematically) to include this gauge transformation into the little group, i.e. instead of Eq. (6) take

$$\mathring{\Lambda} \mathring{p} = \lambda \mathring{p} \quad (11)$$

as defining equation for the little group, where $\mathring{p} = (\varepsilon, 0, 0, 1)$ with $\varepsilon = \pm 1$ and $\lambda > 0$, $\mathring{\Lambda} \in \text{Lor}_{1,3}$. If $\Lambda = \exp(-\frac{1}{2}\omega^{\mu\nu}e_{\mu\nu})$, Eq. (11) yields $\omega^\mu{}_\nu \mathring{p}^\nu = \delta\lambda \mathring{p}^\mu$ or more explicitly

$$\omega_{03} = \delta\lambda\varepsilon, \quad \varepsilon\omega_{01} - \omega_{13} = 0, \quad \varepsilon\omega_{02} - \omega_{23} = 0.$$

Therefore $(\varepsilon^2 - 1)\delta\lambda = 0$, and if $\delta\lambda > 0$, i.e. $\lambda = 1 + \delta\lambda > 1$, the only solutions for Eq. (11) are in case $\varepsilon^2 = 1$. Notice that in case of $E(2)$ as little group one has $\delta\lambda = 0$. The solution of Eq. (11) reads

$$\mathring{\Lambda} = B^{(\varepsilon)}(\vec{\xi}; \lambda, \omega) = \begin{pmatrix} \frac{1}{2}(\frac{1}{\lambda}\xi^2 + \lambda + \frac{1}{\lambda}) & \varepsilon\vec{\xi}^T \text{rot } \omega & \frac{\varepsilon}{2}(-\frac{1}{\lambda}\xi^2 + \lambda - \frac{1}{\lambda}) \\ \frac{\varepsilon}{\lambda}\vec{\xi} & \text{rot } \omega & -\frac{1}{\lambda}\vec{\xi} \\ \frac{\varepsilon}{2}(\frac{1}{\lambda}\xi^2 + \lambda - \frac{1}{\lambda}) & \vec{\xi}^T \text{rot } \omega & \frac{1}{2}(-\frac{1}{\lambda}\xi^2 + \lambda + \frac{1}{\lambda}) \end{pmatrix}, \quad (12)$$

where

$$\vec{\xi} = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}, \quad \text{rot } \omega = \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix}, \quad \xi^2 = \xi_1^2 + \xi_2^2.$$

If one now defines

$$\begin{aligned} B_\lambda^{(\varepsilon)} &= B^{(\varepsilon)}(0; \lambda, 0) = \begin{pmatrix} \frac{1}{2}(\lambda + \frac{1}{\lambda}) & \vec{0}^T & \frac{\varepsilon}{2}(\lambda - \frac{1}{\lambda}) \\ \vec{0} & \mathbb{1}_2 & \vec{0} \\ \frac{\varepsilon}{2}(\lambda - \frac{1}{\lambda}) & \vec{0}^T & \frac{1}{2}(\lambda + \frac{1}{\lambda}) \end{pmatrix}, \\ R_\omega &= B^{(\varepsilon)}(0; 1, \omega) = \begin{pmatrix} 1 & \vec{0}^T & 0 \\ \vec{0} & \text{rot } \omega & \vec{0} \\ 0 & \vec{0}^T & 1 \end{pmatrix}, \\ T_\xi^{(\varepsilon)} &= B^{(\varepsilon)}(\vec{\xi}; 1, 0) = \begin{pmatrix} 1 + \frac{1}{2}\xi^2 & \varepsilon\vec{\xi}^T & -\frac{\varepsilon}{2}\xi^2 \\ \varepsilon\vec{\xi} & \mathbb{1}_2 & -\vec{\xi} \\ \frac{\varepsilon}{2}\xi^2 & \vec{\xi}^T & 1 - \frac{1}{2}\xi^2 \end{pmatrix}, \end{aligned} \quad (13)$$

the general transformation (12) can be written as

$$B^{(\varepsilon)}(\vec{\xi}; \lambda, \omega) = T_{\xi}^{(\varepsilon)} B_{\lambda}^{(\varepsilon)} R_{\omega}.$$

One easily obtains the multiplication table

$$\begin{aligned} B_{\mu}^{(\varepsilon)} B_{\lambda}^{(\varepsilon)} &= B_{\mu\lambda}^{(\varepsilon)} = B_{\lambda}^{(\varepsilon)} B_{\mu}^{(\varepsilon)}, \\ R_{\phi} R_{\omega} &= R_{\phi+\omega} = R_{\omega} R_{\phi}, \\ T_{\eta}^{(\varepsilon)} T_{\xi}^{(\varepsilon)} &= T_{\eta+\xi}^{(\varepsilon)} = T_{\xi}^{(\varepsilon)} T_{\eta}^{(\varepsilon)}, \\ B_{\lambda}^{(\varepsilon)} T_{\xi}^{(\varepsilon)} &= T_{\lambda\xi}^{(\varepsilon)} B_{\lambda}^{(\varepsilon)}, \\ R_{\omega} T_{\xi}^{(\varepsilon)} &= T_{\text{rot } \omega \xi}^{(\varepsilon)} R_{\omega}, \\ B^{(\varepsilon)}(\vec{\xi}; \lambda, \omega) B^{(\varepsilon)}(\vec{\eta}; \mu, \varphi) &= B^{(\varepsilon)}(\vec{\xi} + \lambda \text{rot } \omega \vec{\eta}; \lambda\mu, \omega + \varphi), \\ (B^{(\varepsilon)}(\vec{\xi}; \lambda, \omega))^{-1} &= B^{(\varepsilon)}(-\frac{1}{\lambda} \text{rot}(-\omega) \vec{\xi}; \frac{1}{\lambda}, -\omega). \end{aligned} \tag{14}$$

From the multiplication table (14) it follows that the transformations $B^{(\varepsilon)}(\vec{\xi}; \lambda, \omega)$ form a group $\text{Bor}_{1,3}^{(\varepsilon)} \subset \text{Lor}_{1,3}$ with non-compact parameter space

$$\{\vec{\xi} \in \mathbb{R}_2, 0 \leq \omega \leq \pi, \lambda > 0\}.$$

It can easily be shown that the derived series of commutators \mathcal{D} for $\text{Bor}_{1,3}^{(\varepsilon)}$ ends in the identity id. In fact,

$$\mathcal{D}^2(\text{Bor}_{1,3}^{(\varepsilon)}) = \{\text{id}\}.$$

Actually, $\text{Bor}_{1,3}^{(\varepsilon)}$ is a maximal, solvable and non-compact subgroup of $\text{Lor}_{1,3}$, i.e. the non-compact Borel subgroup of $\text{Lor}_{1,3}$. Moreover, one obtains the Borel decomposition as the semidirect product

$$\mathcal{B}^{(\varepsilon)} \equiv \text{Bor}_{1,3}^{(\varepsilon)} = \mathcal{T}_2^{(\varepsilon)} \rtimes \text{Tor}_{1,3}^{(\varepsilon)}. \tag{15}$$

The set $\mathcal{T}_2^{(\varepsilon)} = \text{Gen}\{T_{\xi}^{(\varepsilon)} : \vec{\xi} \in \mathbb{R}_2\}$ of unipotent elements of $\text{Bor}_{1,3}^{(\varepsilon)}$ is a closed nilpotent subgroup of $\text{Bor}_{1,3}^{(\varepsilon)}$. It contains the subgroup $\mathcal{D}(\text{Bor}_{1,3}^{(\varepsilon)}) = (\text{Bor}_{1,3}^{(\varepsilon)}, \text{Bor}_{1,3}^{(\varepsilon)})$ generated by the

commutators and is normal in $\text{Bor}_{1,3}^{(\varepsilon)}$. $\text{Tor}_{1,3}^{(\varepsilon)} = \text{Bor}_{1,3}^{(\varepsilon)} / \mathcal{T}_2^{(\varepsilon)} = \text{Gen}\{B_\lambda^{(\varepsilon)}, R_\omega : \Lambda > 0; 0 \leq \omega \leq \pi\}$ is the maximal torus in $\text{Bor}_{1,3}^{(\varepsilon)}$ (and in $\text{Lor}_{1,3}$) with dimension $\dim(\text{Bor}_{1,3}^{(\varepsilon)} / \mathcal{T}_2^{(\varepsilon)})$, generated by the semisimple elements $B_\lambda^{(\varepsilon)}$ and R_ω . By the Lie–Kolchin theorem as it is written down later, $\text{Bor}_{1,3}^{(\varepsilon)}$ is upper triangular [6].

At this point a remark of S. Weinberg is of order [8]: “For the case of zero mass there are interesting complications. The little group as Wigner pointed out is a non-semi-simple group, and one must make special remarks about its invariant Abelian subalgebra.” Indeed,

$$\text{Lg } \mathring{p} \sim \text{Abelian}_2 \rtimes \text{Abelian}_2.$$

is the semidirect product (15) of two two-parametric Abelian groups.

2.2 Jordan factorisation

The Jordan factorisation of $M \in \text{GL}_4(\mathbb{R})$ into a semisimple and an unipotent component is given by

$$M = M_u M_s.$$

Since $\text{Bor}_{1,3}$ is solvable, according to the Lie–Kolchin theorem a basis can be chosen with respect to which $B \in \text{Bor}_{1,3}$ can be put into a triangular form

$$B = \begin{pmatrix} * & * & * & * \\ 0 & * & * & * \\ 0 & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} = \begin{pmatrix} 1 & * & * & * \\ 0 & 1 & * & * \\ 0 & 0 & 1 & * \\ 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} = B_u B_s$$

where B_u is unipotent (i.e. all eigenvalues of B_u are 1) and $B_s = \text{diag}(\lambda_0, \lambda_1, \lambda_2, \lambda_3)$ is semisimple. The eigenvalues of B and B_s are identical. In this form, the Jordan decomposition is given by

$$\text{Bor}_{1,3} \subset U_4(\mathbb{R}) \rtimes D_4(\mathbb{R}) \equiv T_4(\mathbb{R}),$$

where $U_4(\mathbb{R})$ is the group of upper triangular unipotent matrices and $D_4(\mathbb{R})$ is the group of invertible diagonal matrices. If B is solvable, then [9]

$$B = \begin{pmatrix} \lambda_0 & * & * & * \\ 0 & \lambda_1 & * & * \\ 0 & 0 & \lambda_2 & * \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix} \Rightarrow B_u = B \begin{pmatrix} \lambda_0 & 0 & 0 & 0 \\ 0 & \lambda_1 & 0 & 0 \\ 0 & 0 & \lambda_2 & 0 \\ 0 & 0 & 0 & \lambda_3 \end{pmatrix}^{-1}$$

and $\det B = 1$. From Ref. [6] we take the following set of theorems, lemma and propositions.

Theorem: Let G be a connected linear algebraic group. Then G contains a Borel subgroup \mathcal{B} , and all other Borel subgroups of G are conjugate to \mathcal{B} . The homogeneous space G/\mathcal{B} is a projective variety.

Lie–Kolchin theorem: If $\pi : G \rightarrow \mathrm{GL}(V)$ is a linear representation of a connected solvable group, then $\pi(G)$ leaves a flag in V invariant, i.e. $\pi(G)$ can be put in triangular form.

Borel Fixpoint Theorem: Let S be a connected, solvable group that acts algebraically on a projective variety X . Then there exists a point $x \in X$ such that

$$\Lambda x = x \quad \text{for all } \Lambda \in S.$$

Therefore, Eq. (11) is reasonable, as the semi-invariant of $\mathrm{Bor}_{1,3}$ in $\mathbb{E}_{1,3}$ is the non-zero vector \mathring{p} spanning the \mathcal{B} -stable line in $\mathbb{E}_{1,3}$ ($x = \mathbb{R}\mathring{p}$).

Lemma: Let V be a \mathbb{C} -vector space of dimension $n > 0$ and S a connected, solvable subgroup of $\mathrm{GL}(V)$. Then there exists a vector $v \in V \setminus \{0\}$ (i.e. $v \neq 0$) such that

$$Sv = \mathbb{C}v.$$

Let \mathcal{B}_s denote the set of semisimple elements of \mathcal{B} and \mathcal{B}_u the set of unipotent elements.

Proposition: If \mathcal{B} is connected and solvable, then the set \mathcal{B}_u is a closed, connected and nilpotent subgroup of \mathcal{B} , containing $\mathcal{D}(G)$ and, therefore, is normal in \mathcal{B} .

From this it follows that the set \mathcal{B}_s is not a closed subgroup of \mathcal{B} because if it would be a subgroup, \mathcal{B} would be nilpotent.

Proposition: Let \mathcal{B} be connected and solvable and let $\mathfrak{b} \equiv \mathcal{L}(\mathcal{B})$ be its Lie algebra. Then the set $\mathcal{L}(\mathcal{B}_u)$ is the set of nilpotent elements of \mathfrak{g} (i.e. $\text{ad}_{\mathfrak{g}}u$ is nilpotent for $u \in \mathcal{L}(\mathcal{B}_u)$).

It is important that for solvable \mathcal{B} the set \mathcal{B}_u of unipotent elements of \mathcal{B} is a connected, closed, normal and nilpotent subgroup of \mathcal{B} , $\mathcal{B}/\mathcal{B}_u$ is a torus, and $\mathcal{D}(\mathcal{B}) \subset \mathcal{B}_u$.

Theorem (Borel): Let \mathcal{B} be a connected, solvable, linear algebraic group. If Tor is a maximal torus of \mathcal{B} , then

$$\mathcal{B} = \mathcal{B}_u \rtimes \text{Tor}.$$

Otherwise, there exists such a torus $\text{Tor} \subset \mathcal{B}$ such that

$$\mathcal{B} = \text{Rad}_u(\mathcal{B}) \rtimes \text{Tor},$$

where $\text{Rad}_u(\mathcal{B})$ is the unipotent radical of \mathcal{B} , leading to the factorisation (15).

The generators of the Borel subgroup in the representation (13) are defined by

$$b_{\mu}^{(\varepsilon)} = \left. \frac{\partial B^{(\varepsilon)}(\omega_{\mu})}{\partial \omega_{\mu}} \right|_{\omega}, \quad (16)$$

where $\omega_0 = \lambda$, $\omega_1 = \xi_1$, $\omega_2 = \xi_2$, $\omega_3 = \omega$ and $\hat{\omega} = (1, 0, 0, 0)$. This yields the Lie algebra $\text{bor}_{1,3}^{(\varepsilon)}$ as basis for the underlying vector space to be generated by

$$\begin{aligned} b_0^{(\varepsilon)} &= \left. \frac{\partial B_{\lambda}^{(\varepsilon)}}{\partial \lambda} \right|_{\lambda=1} = \varepsilon e_{03}, \\ b_1^{(\varepsilon)} &= \left. \frac{\partial T_{\xi}}{\partial \xi_1} \right|_{\vec{\xi}=\vec{0}} = \varepsilon e_{01} + e_{31}, \\ b_2^{(\varepsilon)} &= \left. \frac{\partial T_{\xi}}{\partial \xi_2} \right|_{\vec{\xi}=\vec{0}} = \varepsilon e_{02} + e_{32}, \\ b_3^{(\varepsilon)} &= \left. \frac{\partial R_{\omega}}{\partial \omega} \right|_{\omega=0} = e_{21}. \end{aligned} \quad (16)$$

The commutator relations are $(a, b \in \{1, 2\})$ [10]

$$\begin{aligned} [b_0^{(\epsilon)}, b_a^{(\epsilon)}] &= b_a^{(\epsilon)}, & [b_0^{(\epsilon)}, b_3^{(\epsilon)}] &= 0, \\ [b_3^{(\epsilon)}, b_a^{(\epsilon)}] &= -\epsilon_{3ab} b_b^{(\epsilon)}, & [b_a^{(\epsilon)}, b_b^{(\epsilon)}] &= 0. \end{aligned} \quad (17)$$

The algebra $\text{bor}_{1,3}$ is solvable because

$$[\mathcal{D}(\text{bor}_{1,3}^{(\epsilon)}), \mathcal{D}(\text{bor}_{1,3}^{(\epsilon)})] = \mathcal{D}^2(\text{bor}_{1,3}^{(\epsilon)}) = \{0\}$$

and maximal in $\text{Lor}_{1,3}$, i.e. the Borel algebra of $\text{lor}_{1,3}$. Moreover,

$$\text{bor}_{1,3}^{(\epsilon)} = \mathfrak{t}_2^{(\epsilon)} \rtimes \text{tor}_{1,3}^{(\epsilon)}, \quad (18)$$

where the vectorspace underlying $\mathfrak{t}_2^{(\epsilon)}$ is $\bar{\mathfrak{t}}_2^{(\epsilon)} = \text{span}_{\mathbb{R}}\{b_1^{(\epsilon)}, b_2^{(\epsilon)}\}$ and that of $\text{tor}_{1,3} = \text{car}_{1,3}$ is $\vec{\text{tor}}_{1,3}^{(\epsilon)} = \text{span}_{\mathbb{R}}\{b_0^{(\epsilon)}, b_3^{(\epsilon)}\}$ ($\text{car}_{1,3}$ is the Cartan subalgebra of $\text{lor}_{1,3}$). Therefore, one can conclude that in the massless case the (enlarged) little algebra $\text{bor}_{1,3}^{(\epsilon)}$ is a maximal, non-compact and solvable Lie subalgebra of $\text{lor}_{1,3}$, i.e. the Borel subalgebra, and is the semidirect sum of two abelian algebras $\mathfrak{t}_2^{(\epsilon)}$ and $\text{tor}_{1,3}^{(\epsilon)}$. Notice that there exists no Casimir operator.

In the general case [6], if \mathcal{B} is solvable, its Lie algebra $\mathfrak{b} = \mathcal{L}(\mathcal{B})$ is solvable. Since \mathcal{B}_u is normal in \mathcal{B} , its Lie algebra $\mathfrak{n} = \mathcal{L}(\mathcal{B}_u)$ is an ideal of \mathfrak{b} and \mathfrak{n} is the set of nilpotent elements of \mathfrak{b} . Moreover, since $\mathcal{D}(\mathcal{B}) \subset \mathcal{B}_u$ one has $[\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{n}$.

Theorem: There exists a Lie subalgebra \mathfrak{a} of \mathfrak{b} obeying the conditions

1. \mathfrak{a} is abelian and all its elements are semisimple, i.e. $\mathfrak{a} \subset \mathfrak{b}_s$;
2. as a vector space one has $\mathfrak{b} = \mathfrak{n} \oplus \mathfrak{a}$.

Theorem: Let \mathfrak{b} be algebraic and solvable in $\text{gl}(V)$ and let \mathfrak{n} be the set of nilpotent endomorphisms of V in \mathfrak{b} . If \mathfrak{h} is the maximal commutative Lie subalgebra of \mathfrak{b} consisting of semisimple elements, \mathfrak{b} is the semidirect product of \mathfrak{h} with \mathfrak{n} ,

$$\mathfrak{b} = \mathfrak{n} \rtimes \mathfrak{h}.$$

The existence of Borel algebras follows from the triangular decomposition of the semisimple Lie algebra \mathfrak{g} [6, 11],

$$\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{h} \oplus \mathfrak{n}_-$$

where

$$\mathfrak{n}_+ = \sum_{\alpha \in \phi^+} \mathfrak{g}_\alpha, \quad \mathfrak{n}_- = \sum_{\alpha \in \phi^+} \mathfrak{g}_{-\alpha}$$

and \mathfrak{h} being the Cartan subalgebra of \mathfrak{g} . Then $\mathfrak{b}_\pm = \mathfrak{h} \oplus \mathfrak{n}_\pm$ are maximal solvable Lie subalgebras of \mathfrak{g} , called the Borel subalgebra of \mathfrak{g} relative to \mathfrak{h} . Moreover,

$$\mathfrak{n}_\mathfrak{g}(\mathfrak{b}_\pm) = \mathfrak{b}_\pm \quad \text{and} \quad \mathfrak{z}_\mathfrak{g}(\mathfrak{b}_\pm) = \{0\}$$

where $\mathfrak{n}_\mathfrak{g}$ is the normaliser and $\mathfrak{z}_\mathfrak{g}$ is the centraliser of \mathfrak{b}_\pm in \mathfrak{g} .

2.3 Generators of the Borel subgroup

It can be easily seen that the exponential operation provides the parametrisation of the generic element $B^{(\varepsilon)}(\vec{\xi}; \lambda, \omega) \in \text{Bor}_{1,3}$,

$$\begin{aligned} B_{e^\lambda}^{(\varepsilon)} &= \exp(\tilde{\lambda} b_0^{(\varepsilon)}) = \mathbb{1}_4 + \sinh \tilde{\lambda} b_0^{(\varepsilon)} + (\cosh \tilde{\lambda} - 1)(b_0^{(\varepsilon)})^2 \quad (\lambda = e^{\tilde{\lambda}}), \\ T_\xi^{(\varepsilon)} &= \exp(\vec{\xi} \vec{b}^{(\varepsilon)}) = \mathbb{1}_4 + \vec{\xi} \vec{b}^{(\varepsilon)} + \frac{1}{2}(\vec{\xi} \vec{b}^{(\varepsilon)})^2 = \\ &= \mathbb{1}_4 + \xi_1 b_1^{(\varepsilon)} + \xi_2 b_2^{(\varepsilon)} + \frac{1}{2} \xi_1^2 (b_1^{(\varepsilon)})^2 + \frac{1}{2} \xi_2^2 (b_2^{(\varepsilon)})^2, \\ R_\omega &= \exp(\omega b_3^{(\varepsilon)}) = \mathbb{1}_4 + \sin \omega b_3^{(\varepsilon)} + (1 - \cos \omega)(b_3^{(\varepsilon)})^2, \end{aligned} \tag{19}$$

The general element of $\text{bor}_{1,3}^{(\varepsilon)}$ can be written as

$$Y = y^\mu b_\mu = \begin{pmatrix} 0 & \varepsilon y^1 & \varepsilon y^2 & \varepsilon y^0 \\ \varepsilon y^1 & 0 & -y^3 & -y^1 \\ \varepsilon y^2 & y^3 & 0 & -y^2 \\ \varepsilon y^0 & y^1 & y^2 & 0 \end{pmatrix}. \tag{20}$$

The adjoint representation $\text{ad } Y$ in the basis $\{b_1, b_2, b_0, b_3\}$ of the semidirect sum \rtimes is calculated to be

$$\text{ad } Y (= \text{Reg } Y) = \begin{pmatrix} y^0 & -y^3 & -y^1 & y^2 \\ y^3 & y^0 & -y^2 & -y^1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \quad (21)$$

From the secular equation $\det(\text{ad } Y - \lambda) = (-\lambda)^2((y^0 - \lambda)^2 + (y^3)^2)$ it follows that

$$\text{spec}(\text{ad } Y) = \{0, 0, \lambda_3 = y^0 - iy^3, \lambda_4 = y^0 + iy^3\}. \quad (22)$$

The eigenvalue problem yields two eigenfunctions Z_3 and Z_4 ,

$$\begin{aligned} \text{ad } Y(Z_3) = \lambda_3 Z_3 &\Rightarrow Z_3 = \frac{i}{2}(b_1^{(\varepsilon)} + ib_2^{(\varepsilon)}), \\ \text{ad } Y(Z_4) = \lambda_4 Z_4 &\Rightarrow Z_4 = -\frac{i}{2}(b_1^{(\varepsilon)} - ib_2^{(\varepsilon)}), \end{aligned} \quad (23)$$

i.e. $Z_3, Z_4 \in \mathfrak{t}_2^{(\varepsilon)}$. In the special case $Y = y^0 b_0^{(\varepsilon)} + y^3 b_3^{(\varepsilon)}$ one obtains

$$\begin{aligned} [y^0 b_0 + y^3 b_3, Z_3] &= (y^0 - iy^3)Z_3, \\ [y^0 b_0 + y^3 b_3, Z_4] &= -(y^0 + iy^3)Z_4. \end{aligned} \quad (24)$$

Two cases for Y are important, namely $t_0^{(\varepsilon)} = \frac{1}{2}(b_0^{(\varepsilon)} + ib_3^{(\varepsilon)})$ and $u_0^{(\varepsilon)} = \frac{1}{2}(b_0^{(\varepsilon)} - ib_3^{(\varepsilon)})$.

Combining in pairs with $t_+^{(\varepsilon)} = Z_3$ and $u_+^{(\varepsilon)} = Z_4$,

$$\begin{aligned} t_0^{(\varepsilon)} &= \frac{1}{2}(b_0^{(\varepsilon)} + ib_3^{(\varepsilon)}), & t_+^{(\varepsilon)} &= \frac{i}{2}(b_1^{(\varepsilon)} + ib_2^{(\varepsilon)}), \\ u_0^{(\varepsilon)} &= \frac{1}{2}(b_0^{(\varepsilon)} - ib_3^{(\varepsilon)}), & u_+^{(\varepsilon)} &= -\frac{i}{2}(b_1^{(\varepsilon)} - ib_2^{(\varepsilon)}), \end{aligned} \quad (25)$$

one obtains the commutation relations

$$[t_0^{(\varepsilon)}, t_+^{(\varepsilon)}] = t_+^{(\varepsilon)}, \quad [u_0^{(\varepsilon)}, u_+^{(\varepsilon)}] = u_+^{(\varepsilon)}, \quad [t_{0,+}^{(\varepsilon)}, u_{0,+}^{(\varepsilon)}] = 0. \quad (26)$$

The elements

$$\begin{aligned} t_0^{(\varepsilon)} &= \frac{1}{2}(b_0^{(\varepsilon)} + ib_3^{(\varepsilon)}) = \frac{1}{2}(\varepsilon e_{03} + ie_{21}) = -iJ_3^{(\varepsilon)}, \\ t_+^{(\varepsilon)} &= \frac{i}{2}(b_1^{(\varepsilon)} + ib_2^{(\varepsilon)}) = \frac{1}{2}(i\varepsilon e_{01} - \varepsilon e_{02} + ie_{31} - e_{32}) = J_+^{(\varepsilon)} \end{aligned} \quad (27)$$

generate the minimal solvable algebra $\text{sol}_2^{(\varepsilon)}(t)$ with underlying vector space given by $\text{span}_{\mathbb{R}}\{t_0^{(\varepsilon)}, t_+^{(\varepsilon)}\}$. Similarly,

$$\begin{aligned} u_0^{(\varepsilon)} &= \frac{1}{2}(b_0^{(\varepsilon)} - ib_3^{(\varepsilon)}) = \frac{1}{2}(\varepsilon e_{03} - ie_{21}) = iK_3^{(\varepsilon)}, \\ u_+^{(\varepsilon)} &= -\frac{i}{2}(b_1^{(\varepsilon)} - ib_2^{(\varepsilon)}) = -\frac{1}{2}(i\varepsilon e_{01} + \varepsilon e_{02} + ie_{31} + e_{32}) = K_-^{(\varepsilon)} \end{aligned} \quad (28)$$

generate the algebra $\text{sol}_2^{(\varepsilon)}(u)$, and since $[\text{sol}_2^{(\varepsilon)}(t), \text{sol}_2^{(\varepsilon)}(u)] = 0$, one obtains the decomposition

$$\text{bor}_{1,3}^{(\varepsilon)*} = \text{sol}_2^{(\varepsilon)}(t) \boxplus \text{sol}_2^{(\varepsilon)}(u), \quad (29)$$

where \boxplus is the Kronecker sum, $A \boxplus B = A \otimes \mathbb{1} + \mathbb{1} \otimes B$.

3 Representations

Every representation of $\text{lor}_{1,3}$ defines a particular representation of the subalgebra $\text{bor}_{1,3}$. Of course, not all the representations are of that kind but those defined by $\text{lor}_{1,3}$ are of great importance because the classification of particles is determined by their Lorentz transformation properties according to Eqs. (1) and (2). More precisely, the common eigenvectors of the representation space of the solvable algebra $\text{bor}_{1,3}$ are the possible helicity states of the particle.

Lemma: Let \mathfrak{g} be a solvable algebra and $\mathfrak{g} \rightarrow \Gamma(\mathfrak{g})$ be a representation on a finite-dimensional vector space V . Then

1. there exists a vector $v \in V$ which is a simultaneous eigenvector for all of $\Gamma(\mathfrak{g})$,
2. there exists a basis of V with respect to which all elements of $\Gamma(\mathfrak{g})$ are represented by upper triangular matrices.

Notice that the common eigenvector is determined by all the elements of $\Gamma(\mathfrak{g})$, i.e. in our case $\mathfrak{g} = \text{bor}_{1,3}$ there is no need to assume $\Gamma(\mathfrak{t}_2^{(\varepsilon)}) = 0$. In the complex spaces

$$\text{span}_{\mathbb{C}}\{e_{(\mu)} : e_{(\mu)}^\rho = \eta^\rho{}_\mu = \delta_{\rho\mu}\}_{0,}^3,$$

the eigenvectors of the solvable algebra $\text{sol}_2^{(\varepsilon)}(t)$ are

$$\ell_0^{(\varepsilon)} = \varepsilon e_{(0)} + e_{(3)} = (\varepsilon, 0, 0, 1)^T, \quad \ell_1 = e_{(1)} + ie_{(2)} = (0, 1, i, 0)^T.$$

Indeed,

$$\begin{aligned} t_0^{(\varepsilon)} \ell_0^{(\varepsilon)} &= \frac{1}{2} \ell_0^{(\varepsilon)}, & t_+^{(\varepsilon)} \ell_0^{(\varepsilon)} &= 0, \\ t_0^{(\varepsilon)} \ell_1 &= \frac{1}{2} \ell_1, & t_+^{(\varepsilon)} \ell_1 &= 0. \end{aligned} \quad (30i)$$

Accordingly, the eigenvectors of $\text{sol}_2(u)$ are $\ell_0^{(\varepsilon)}$ and $\ell_2 = e_{(1)} - ie_{(2)} = (0, 1, -i, 0)^T$, where

$$\begin{aligned} u_0^{(\varepsilon)} \ell_0^{(\varepsilon)} &= \frac{1}{2} \ell_0^{(\varepsilon)}, & u_+^{(\varepsilon)} \ell_0^{(\varepsilon)} &= 0, \\ u_0^{(\varepsilon)} \ell_2 &= \frac{1}{2} \ell_2, & u_+^{(\varepsilon)} \ell_2 &= 0. \end{aligned} \quad (30ii)$$

Since

$$\begin{aligned} u_0^{(\varepsilon)} \ell_1 &= -\frac{1}{2} \ell_1, & u_+^{(\varepsilon)} \ell_1 &= -i \ell_0^{(\varepsilon)}, \\ t_0^{(\varepsilon)} \ell_2 &= -\frac{1}{2} \ell_2, & t_+^{(\varepsilon)} \ell_2 &= i \ell_0^{(\varepsilon)} \end{aligned}$$

and $\text{bor}_{1,3}^* = \text{sol}_2(t) \boxplus \text{sol}_2(u)$, the subspace $\text{span}_{\mathbb{C}}\{\ell_0^{(\varepsilon)}, \ell_1, \ell_2\}$ is invariant under the action of $\text{bor}_{1,3}$. However, the vector $\ell_0^{(\varepsilon)}$ is already the defining vector for $\text{bor}_{1,3}$ (cf. Eq. (11)). Therefore, there are two helicity states ℓ_1 and ℓ_2 relative to $\ell_0^{(\varepsilon)}$. More precisely, using the two components \vec{D} and \vec{B} of the Lorentz group defined in Appendix A, the defining Eq. (11) yields the conditions

$$D_3 \ell_0^{(\varepsilon)} = 0, \quad B_3 \ell_0^{(\varepsilon)} = 1 \ell_0^{(\varepsilon)}, \quad (30iii)$$

ℓ_1 is called right-handed with respect to $\ell_0^{(\varepsilon)}$, and ℓ_2 is called left-handed with respect to $\ell_0^{(\varepsilon)}$. The value 1 in Eq. (30iii) may be considered as helicity 1 (not spin because there is no rotation SO_3). The conditions (30iii) are equivalent to

$$t_0^{(\varepsilon)} \ell_0^{(\varepsilon)} = u_0^{(\varepsilon)} \ell_0^{(\varepsilon)} = \frac{1}{2} \ell_0^{(\varepsilon)}, \quad t_+^{(\varepsilon)} \ell_0^{(\varepsilon)} = u_+^{(\varepsilon)} \ell_0^{(\varepsilon)} = 0.$$

3.1 The Chevalley basis

Depending to the sign of $\varepsilon = \pm 1$, the elements of $\text{sol}_2(t)$ and $\text{sol}_2(u)$ can be represented by elements of the fundamental sl_2 (or Chevalley) basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}.$$

According to the commutator relations, for $\varepsilon = +1$ one can represent $t_0^{(\varepsilon)}$ and $u_0^{(\varepsilon)}$ by $\mp \varepsilon i h/2$, $t_+^{(\varepsilon)}$ and $u_+^{(\varepsilon)}$ by ie and if – or vice versa for $\varepsilon = -1$ (cf. Eqs. (A37,A38)). The two solvable algebras $\text{sol}_2(e) = \text{span}_{\mathbb{R}}\{h, ie\}$ and $\text{sol}_2(f) = \text{span}_{\mathbb{R}}\{h, if\}$ are representations of the algebras $\text{sol}_2(t)$ and $\text{sol}_2(u)$, respectively,

$$\begin{aligned} t_0^{(\varepsilon)} &= \left(\frac{1+\varepsilon}{4} h \right) \boxplus \left(\frac{1-\varepsilon}{2} h \right) = -iJ_3^{(\varepsilon)}, & t_+^{(\varepsilon)} &= \left(\frac{1+\varepsilon}{2} ie \right) \boxplus \left(\frac{1-\varepsilon}{2} ie \right) = J_+^{(\varepsilon)}, \\ u_0^{(\varepsilon)} &= - \left(\left(\frac{1-\varepsilon}{4} h \right) \boxplus \left(\frac{1+\varepsilon}{4} h \right) \right) = iK_3^{(\varepsilon)}, & u_+^{(\varepsilon)} &= \left(\frac{1-\varepsilon}{2} if \right) \boxplus \left(\frac{1+\varepsilon}{2} if \right) = K_-^{(\varepsilon)}. \end{aligned} \quad (31)$$

This can be generalised to arbitrary representations as for instance the subrepresentations of the representation (k, l) of $\text{lor}_{1,3}$. The general representation (k, l) is given by the rule

$$\begin{aligned} \pi^{(k,l)} : \text{bor}_{1,3}^{(\varepsilon)*} &= \text{sol}_2^{(\varepsilon)}(e) \boxplus \text{sol}_2^{(\varepsilon)}(f) \rightarrow \\ &\rightarrow \pi^{(k,l)}(\text{bor}_{1,3}^{(\varepsilon)*}) = \pi^{(k)}(\text{sol}_2^{(\varepsilon)}(e)) \boxplus \pi^{(l)}(\text{sol}_2^{(\varepsilon)}(f)), \end{aligned} \quad (32)$$

where

$$\text{sol}_2^{(\varepsilon)}(e) = \frac{1+\varepsilon}{2} \text{sol}_2(e) \boxplus \frac{1-\varepsilon}{2} \text{sol}_2(e), \quad \text{sol}_2^{(\varepsilon)}(f) = \frac{1-\varepsilon}{2} \text{sol}_2(f) \boxplus \frac{1+\varepsilon}{2} \text{sol}_2(f). \quad (33)$$

By virtue of this construction one obtains

$$\begin{aligned} \pi^{(k,l)}(t_0^{(\varepsilon)}) &= - \left(\frac{i(1+\varepsilon)}{4} \pi^{(k)}(h) \boxplus \frac{i(1-\varepsilon)}{4} \pi^{(l)}(h) \right) = \\ &= \begin{cases} -\frac{i}{2} \pi^{(k)}(h) \otimes \mathbb{1}_l & \text{for } \varepsilon = +1, \\ -\mathbb{1}_k \otimes \frac{i}{2} \pi^{(l)}(h) & \text{for } \varepsilon = -1, \end{cases} \end{aligned}$$

$$\begin{aligned}
\pi^{(k,l)}(t_+^{(\varepsilon)}) &= \frac{i(1+\varepsilon)}{2}\pi^{(k)}(e) \boxplus \frac{i(1-\varepsilon)}{2}\pi^{(l)}(e) = \\
&= \begin{cases} i\pi^{(k)}(e) \otimes \mathbb{1}_l & \text{for } \varepsilon = +1, \\ \mathbb{1}_k \otimes i\pi^{(l)}(e) & \text{for } \varepsilon = -1, \end{cases} \\
\pi^{(k,l)}(u_0^{(\varepsilon)}) &= \frac{i(1-\varepsilon)}{4}\pi^{(k)}(h) \boxplus \frac{i(1+\varepsilon)}{4}\pi^{(l)}(h) = \\
&= \begin{cases} \mathbb{1}_k \otimes \frac{i}{2}\pi^{(l)}(h) & \text{for } \varepsilon = +1, \\ \frac{i}{2}\pi^{(k)}(h) \otimes \mathbb{1}_l & \text{for } \varepsilon = -1, \end{cases} \\
\pi^{(k,l)}(u_+^{(\varepsilon)}) &= \frac{i(1-\varepsilon)}{2}\pi^{(k)}(f) \boxplus \frac{i(1+\varepsilon)}{2}\pi^{(l)}(f) = \\
&= \begin{cases} \mathbb{1}_k \otimes i\pi^{(l)}(f) & \text{for } \varepsilon = +1, \\ i\pi^{(k)}(f) \otimes \mathbb{1}_l & \text{for } \varepsilon = -1, \end{cases}
\end{aligned} \tag{34}$$

where

$$\begin{aligned}
\pi^{(k)}(h)|k, m\rangle &= 2m|k, m\rangle, \\
\pi^{(k)}(e)|k, m\rangle &= \rho_{(m)}^{(k)}|k, m+1\rangle, \\
\pi^{(k)}(f)|k, m\rangle &= \rho_{(-m)}^{(k)}|k, m-1\rangle,
\end{aligned} \tag{35}$$

where $\rho_{(m)}^{(k)} = \sqrt{(k-m)(k+m+1)}$. In the representation of $\pi^{(k,l)}$ by direct products,

$$\{|k, l; m_k, m_l\rangle = |k, m_k\rangle \otimes |l, m_l\rangle : -k \leq m_k \leq k; -l \leq m_l \leq l\}$$

one obtains common eigenvectors for $\text{sol}_2(t)$ given by

$$\pi^{(k,l)}(t_0^{(+)})|k, l; k, m_l\rangle = k|k, l; k, m_l\rangle, \quad \pi^{(k,l)}(t_+^{(+)})|k, l; k, m_l\rangle = 0 \tag{36}$$

with $m_l = -l, -l+1, \dots, l$, and common eigenvectors for $\text{sol}_2(u)$ given by

$$\pi^{(k,l)}(u_0^{(+)})|k, l; m_k, -l\rangle = l|k, l; m_k, -l\rangle, \quad \pi^{(k,l)}(u_+^{(+)})|k, l; m_k, -l\rangle = 0 \tag{37}$$

with $m_k = -k, -k+1, \dots, k$.

3.2 Resolution of the solvable group

$\text{sol}_2(e)$ is a Lie algebra of orientation-conserving affine translations Aff_1 . The underlying topological space $\mathbb{R} \times \mathbb{R}_+$ for the corresponding Lie group $\text{Sol}_2(e)$ is simply connected and

open in the plane \mathbb{R}^2 . As a general element of this group can be represented by

$$S(\beta, \alpha) = \begin{pmatrix} e^\alpha & \beta \\ 0 & 1 \end{pmatrix}$$

($\alpha, \beta \in \mathbb{R}$), the geometrical space on which this group acts is the real line,

$$\text{Sol}_2(e) \ni S(\beta, \alpha) : \mathbb{R}^1 \ni x \rightarrow e^\alpha x + \beta \in \mathbb{R}^1.$$

Because of

$$S(\beta, \alpha)S(\beta', \alpha') = \begin{pmatrix} e^{\alpha+\alpha'} & e^\alpha \beta' + \beta \\ 0 & 1 \end{pmatrix} = S(e^\alpha \beta' + \beta, \alpha + \alpha'),$$

the solvable group as a semidirect product of two abelian groups,

$$\text{Sol}_2(e) = \mathbb{R} \rtimes \mathbb{R}_+.$$

Therefore, Sol_2 is

1. locally compact
2. simply connected
3. minimal non-abelian
4. non-compact
5. non-semisimple, i.e. solvable,
6. non-unimodular.

Via the exponential mapping, these two parts are generated by e and $h_+ = \frac{1}{2}(\mathbb{1} + h)$. Therefore, one can write $\text{sol}_2(e) = \mathbb{R}e \rtimes \mathbb{R}h_+$. A similar consideration leads to $\text{sol}_2(f) = \mathbb{R}f \rtimes \mathbb{R}h_-$ where $h_- = \frac{1}{2}(\mathbb{1} - h)$. Moreover, $\text{sol}_2(e)$ and $\text{sol}_2(f)$ are related to the Cartan involution. Therefore, we end up with the amusing two-fold decomposition

$$\text{bor}_{1,3}^* = (\mathbb{R}e \rtimes \mathbb{R}h_+) \boxplus (\mathbb{R}f \rtimes \mathbb{R}h_-)$$

where

$$h_+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h_- = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.$$

It is important to note that the Kronecker sum structure is imposed by semisimplicity because $\mathfrak{bor}_{1,3}$ is a real form of $\mathfrak{so}_4(\mathbb{C})$, while the semidirect product structure is caused by solvability.

3.3 Weinberg's ansatz

From Steven Weinberg we adopt the following statements which we subsume under the keyword of “Weinberg's Ansatz” [3]:

1. If a massless particle is equal to it's antiparticle, it is described by the irreducible representation (k, k) of the proper Lorentz group.
2. If a massless particle is not equal to it's antiparticle, the particle is described by the irreducible representation $(k, 0)$ of the proper Lorentz group, the antiparticle is described by the irreducible representation $(0, k)$ of the proper Lorentz group.

Note that the massless particle is defined via the Borel subgroup by the irreducible representation of the proper Lorentz group without necessity to introduce parity separately.

For the representation (k, k) , from Eq. (36) one obtains the helicity states associated with $\mathfrak{sol}_2^{(+)}(e)$,

$$|k, k; , k, -k + p\rangle, \quad p = 0, 1, 2, \dots, 2k,$$

and from Eq. (37) one obtains the helicity states associated with $\mathfrak{sol}_2^{(+)}(f)$,

$$|k, k; k - p, -k\rangle, \quad p = 0, 1, 2, \dots, 2k.$$

However, the condition (30iii) excludes the state $|k, k; k, -k\rangle$, since

$$\begin{aligned} D_3^{(k,k)}|k, k; k, -k\rangle &= i(k - k)|k, k; k, -k\rangle = 0, \\ B_3^{(k,k)}|k, k; k, -k\rangle &= 2k|k, k; k, -k\rangle. \end{aligned}$$

Therefore, for the particle ($\varepsilon = +1$) with zero mass and helicity $\lambda = 2k$ one obtains the $4k$ helicity states

$$|k, k; k, -k + p\rangle, \quad |k, k; k - p, -k\rangle, \quad p = 1, 2, \dots, 2k \quad (38)$$

relative to the central state $|k, k; k, -k\rangle$ which is determined by the conditions

$$\begin{aligned} t_0^{(+)}|k, k; k, -k\rangle &= u_0^{(+)}|k, k; k, -k\rangle = k|k, k; k, -k\rangle, \\ t_+^{(+)}|k, k; k, -k\rangle &= u_+^{(+)}|k, k; k, -k\rangle = 0. \end{aligned} \quad (39)$$

On setting $\varepsilon = -1$ in Eq. (34), the helicity states become

$$|k, k; -k + p, k\rangle, \quad |k, k; -k, k - p\rangle, \quad p = 1, 2, \dots, 2k$$

The central state $|k, k; -k, k\rangle$ is defined by the conditions

$$\begin{aligned} t_0^{(-)}|k, k; -k, k\rangle &= u_0^{(-)}|k, k; -k, k\rangle = k|k, k; -k, k\rangle, \\ t_+^{(-)}|k, k; -k, k\rangle &= u_+^{(-)}|k, k; -k, k\rangle = 0. \end{aligned} \quad (40)$$

Notice that for the vector case $(\frac{1}{2}, \frac{1}{2})$ and $\varepsilon = +1$ the helicity states are given by

$$|\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, \frac{1}{2}\rangle, \quad |\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, -\frac{1}{2}\rangle$$

relative to the central state $|\frac{1}{2}, \frac{1}{2}; \frac{1}{2}, -\frac{1}{2}\rangle$. For $\varepsilon = -1$ the helicity states are the same, but now relative to the central state $|\frac{1}{2}, \frac{1}{2}; -\frac{1}{2}, \frac{1}{2}\rangle$. Therefore, the two polarisations of the photon as basic quantities in physics are determined by the proper Lorentz group $\text{Lor}_{1,3}$ without taking refuge to the parity.

As a further example, the massless particle of helicity $\lambda = 2$ is associated with the representation $(1, 1)$ and has 4 helicity states relative to the state $|1, 1; 1, -1\rangle$: two right-handed states $|1, 1; 1, 1\rangle, |1, 1; 1, 0\rangle$ and two left-handed states $|1, 1; 0, -1\rangle, |1, 1; -1, -1\rangle$.

3.4 Weyl equations [12]

The Weyl equations are of the type $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$. The representation $(\frac{1}{2}, 0)$ is defined by the commutative diagram

$$\begin{array}{ccc} \text{Lor}_{1,3} \ni \Lambda : & \mathbb{E}_{1,3} \ni p^\mu & \longrightarrow (\Lambda p)^\mu = \Lambda^\mu{}_\nu p^\nu \\ & \downarrow \downarrow \sigma & \downarrow \sigma \\ \text{SL}_2(\mathbb{C}) \ni \pm A_\Lambda : & \mathbb{H}_2 \in \sigma(p) = \sigma_\mu p^\mu & \longrightarrow A_\Lambda \sigma(p) A_\Lambda^\dagger \end{array}$$

where the commutativity of the diagram results in

$$A_\Lambda \sigma(p) A_\Lambda^\dagger = \sigma(\Lambda p) = \sigma_\mu \Lambda^\mu{}_\nu p^\nu.$$

Using $\tilde{\sigma} = (\mathbb{1}; -\vec{\sigma})$, one obtains

$$A_\Lambda = \frac{\Lambda^{\mu\nu} \sigma_\mu \tilde{\sigma}_\nu}{2 \text{tr}(A_\Lambda^\dagger)}, \quad \Lambda_{A\nu}^\mu = \frac{1}{2} \text{tr}(\sigma_\mu A_\Lambda \sigma_\nu A_\Lambda^\dagger). \quad (41)$$

Using the exponential representations $\Lambda = \exp(-\frac{1}{2}\omega^{\mu\nu}e_{\mu\nu})$ and $A_\Lambda = \exp(-\frac{1}{2}\omega^{\mu\nu}m_{\mu\nu})$, one obtains a relation between the generators,

$$m_{\mu\nu} = \frac{1}{4}(e_{\mu\nu})^{\alpha\beta} \sigma_\alpha \tilde{\sigma}_\beta = \begin{cases} m_{kl} = \frac{i}{2} \epsilon_{kl}{}^j \sigma_j \\ m_{0j} = \frac{1}{2} \sigma_j \end{cases}$$

One can obtain the transformation rule of the Weyl spinor by looking at the commutative diagram

$$\begin{array}{ccc} \psi_R : & \mathbb{E}_{1,3} \ni x & \longrightarrow \psi_R(x) \\ U(\Lambda) \downarrow & \downarrow \Lambda & \downarrow A_\Lambda \\ U(\Lambda) \psi_R : & \mathbb{E}_{1,3} \ni \Lambda x & \longrightarrow (U(\Lambda) \psi_R)(\Lambda x) \end{array}$$

implying $(U(\Lambda) \psi_R)(\Lambda x) = T^{(1/2,0)} \psi_R(x) = A_\Lambda \psi_R(x)$. The Weyl equation

$$\tilde{\sigma}_\mu \partial^\mu \psi_R(x) = 0$$

is given in momentum space by $\tilde{\sigma}_\mu p^\mu \psi_R(p) = 0$. For the standard vector $\overset{\circ}{p} = (\varepsilon, 0, 0, 1)^T$ this equation reduces to

$$\sigma_3 \ell_R^{(\varepsilon)} = \varepsilon \ell_R^{(\varepsilon)}, \quad (42)$$

having the solutions $\ell_R^{(\varepsilon)} = (1 + \varepsilon)a\ell_1 + (1 - \varepsilon)b\ell_2$ with $\ell_1 = (1, 0)^T$ and $\ell_2 = (0, 1)^T$. The Borel algebra $\text{bor}_{1,3}^{(\varepsilon)}(\frac{1}{2}, 0)$ can be expressed as

$$\begin{aligned} b_0^{(\varepsilon)}(\tfrac{1}{2}, 0) &= \frac{\varepsilon}{2}h, & b_1^{(\varepsilon)}(\tfrac{1}{2}, 0) &= \frac{1 + \varepsilon}{2}e - \frac{1 - \varepsilon}{2}f, \\ b_3^{(\varepsilon)}(\tfrac{1}{2}, 0) &= -\frac{i}{2}h, & b_2^{(\varepsilon)}(\tfrac{1}{2}, 0) &= -\frac{i(1 + \varepsilon)}{2}e - \frac{i(1 - \varepsilon)}{2}f. \end{aligned} \quad (43)$$

The algebras $\text{sol}_2^{(\varepsilon)}(e)$ and $\text{sol}_2^{(\varepsilon)}(f)$ have the form

$$\begin{aligned} \text{sol}_2^{(\varepsilon)}(e) &= \left\{ t_0^{(\varepsilon)}(\tfrac{1}{2}, 0) = \frac{1 + \varepsilon}{4}h, \quad t_+^{(\varepsilon)}(\tfrac{1}{2}, 0) = \frac{i(1 + \varepsilon)}{2}e \right\}, \\ \text{sol}_2^{(\varepsilon)}(f) &= \left\{ u_0^{(\varepsilon)}(\tfrac{1}{2}, 0) = -\frac{1 - \varepsilon}{4}h, \quad u_+^{(\varepsilon)}(\tfrac{1}{2}, 0) = \frac{i(1 - \varepsilon)}{2}e \right\}. \end{aligned}$$

The common eigenvector for $\text{sol}_2^{(\varepsilon)}(e)$ is $\ell_1 = (1, 0)^T$,

$$t_0^{(\varepsilon)}(\tfrac{1}{2}, 0)\ell_1 = \frac{1 + \varepsilon}{4}\ell_1, \quad t_+^{(\varepsilon)}(\tfrac{1}{2}, 0)\ell_1 = 0,$$

and for $\text{sol}_2^{(\varepsilon)}(f)$ one obtains $\ell_2 = (0, 1)^T$,

$$u_0^{(\varepsilon)}(\tfrac{1}{2}, 0)\ell_2 = -\frac{1 - \varepsilon}{4}\ell_2, \quad u_+^{(\varepsilon)}(\tfrac{1}{2}, 0)\ell_2 = 0.$$

Therefore, the eigenvector of $\text{bor}_{1,3}^{(\varepsilon)}(\frac{1}{2}, 0)$ is exactly equal to the solution of Weyl's equation, where ℓ_1 is right handed and ℓ_2 is left handed. Notice that in case of the irreducible representation $(\frac{1}{2}, 0)$ of the proper Lorentz group there exists only one single solution, i.e. one helicity state $\lambda = \frac{1}{2}$.

More generally, the representation space of the Lorentz representation $(k, 0)$ is given by

$$V^{(+)}(k, 0) = \text{span}_{\mathbb{C}}\{|k, 0; m, 0\rangle : m = -k, -k + 1, \dots, k\}.$$

The action of $\text{bor}_{1,3}^{(+)}$ on $V^{(+)}(k, 0)$ can be written as

$$\begin{aligned} t_0^{(+)}|k, 0; m, 0\rangle &= m|k, 0; m, 0\rangle, & u_0^{(+)}|k, 0; m, 0\rangle &= 0, \\ t_+^{(+)}|k, 0; m, 0\rangle &= i\rho_{(m)}^{(k)}|k, 0; m + 1, 0\rangle, & u_+^{(+)}|k, 0; m, 0\rangle &= 0. \end{aligned}$$

Therefore, there exists only a single eigenvector $|k, 0; k, 0\rangle \in V^{(+)}(k, 0)$ of the Borel algebra $\text{bor}_{1,3}^{(+)}(k, 0)$, i.e. one a single helicity state with

$$\begin{aligned} t_0^{(+)}|k, 0; k, 0\rangle &= k|k, 0; k, 0\rangle, \\ t_+^{(+)}|k, 0; k, 0\rangle &= 0, \\ u_0^{(+)}|k, 0; k, 0\rangle &= u_+^{(+)}|k, 0; k, 0\rangle = 0. \end{aligned}$$

For $\varepsilon = -1$ the single helicity state can be written as $|k, 0; -k, 0\rangle$ with

$$\begin{aligned} t_0^{(-)}|k, 0; -k, 0\rangle &= t_+^{(-)}|k, 0; -k, 0\rangle = 0, \\ u_0^{(-)}|k, 0; -k, 0\rangle &= k|k, 0; -k, 0\rangle, \\ u_+^{(-)}|k, 0; -k, 0\rangle &= 0. \end{aligned}$$

In the irreducible case $(0, k)$ (and $\varepsilon = +1$) the representation space reads

$$V^{(+)}(0, k) = \text{span}_{\mathbb{C}}\{|0, k; 0, m\rangle : m = -k, -k+1, \dots, k\},$$

and the action of $\text{bor}_{1,3}^{(+)}(0, k)$ is given by

$$\begin{aligned} t_0^{(+)}|0, k; 0, m\rangle &= 0, & u_0^{(+)}|0, k; 0, m\rangle &= -m|0, k; 0, m\rangle, \\ t_+^{(+)}|0, k; 0, m\rangle &= 0, & u_+^{(+)}|0, k; 0, m\rangle &= i\rho_{(-m)}^{(k)}|0, k; 0, m-1\rangle. \end{aligned}$$

The only eigenvector of $\text{bor}_{1,3}^{(+)}(0, k)$ is $|0, k; 0, -k\rangle$ with

$$\begin{aligned} t_0^{(+)}|0, k; 0, -k\rangle &= t_+^{(+)}|0, k; 0, -k\rangle = 0, \\ u_0^{(+)}|0, k; 0, -k\rangle &= k|0, k; 0, -k\rangle, \\ u_+^{(+)}|0, k; 0, -k\rangle &= 0. \end{aligned}$$

For $\varepsilon = -1$ one obtains the action

$$\begin{aligned} t_0^{(-)}|0, k; 0, m\rangle &= m|0, k; 0, m\rangle, & u_0^{(-)}|0, k; 0, m\rangle &= 0, \\ t_+^{(-)}|0, k; 0, m\rangle &= i\rho_{(m)}^{(k)}|0, k; 0, m+1\rangle, & u_+^{(-)}|0, k; 0, m\rangle &= 0 \end{aligned}$$

and the only eigenvector $|0, k; 0, k\rangle$.

4 Conclusions

Turning back to the title of our paper, we used the word “nophysics”. In the end, this term should be explained. On the one hand, we can take it as an abbreviation, namely

$$\text{nophysics} = \text{new old physics}$$

where “old” standas for Pauli’s massless neutrino hypothesis, “new” for the solvability as symmetry applicable to massless particles. On the other hand, “no” can also stand for

- no semisimple (instead, solvable) solution,
- no abelian (instead, the minimal non-abelian) solution,
- no compact (but locally compact) solution,
- no unimodular (instead, non-unimodular) solution,
- no Killing form (because the Cartan–Killing metric tensor is identically zero on the derived algebra), and
- no Casimir invariant found.

If semisimplicity, as we are being used to it, stands for “yes”, solvability represents “no”. However, in our opinion, physics as it should be treated is semisimple as well as solvable, providing a good perspective for our view at the phenomenon of mass.

Taking solvability as the internal symmetry of massless particles, the photon of helicity $\lambda = 1$ is represented by $(\frac{1}{2}, \frac{1}{2})$, and Pauli’s neutrino and antineutrino of helicity $\lambda = \frac{1}{2}$ by $(\frac{1}{2}, 0)$ and $(0, \frac{1}{2})$, respectively. Finally, we might ask about the graviton. If the graviton is identical to its own antiparticle, according to Weinberg’s ansatz it is represented by $(1, 1)$ with helicity $\lambda = 2$. However, if there are graviton and antigraviton, these are represented by $(1, 0)$ and $(0, 1)$ with helicity $\lambda = 1$.

Finally, accepting the Borel algebra as symmetry algebra for massless particles, it is reasonable to develop a Yang–Mills theory for solvable groups, treating the elementary algebras \mathfrak{su}_2 and \mathfrak{sol}_2 on the same footing. Indeed, while the algebra \mathfrak{su}_2 generates semisimple algebras via the Cartan matrix, the solvable algebras are constructed gradually as semidirect sums of abelian algebras. Moreover, the solvable gauge will more immediately generate the abelian gauge of the field theory of electromagnetism, according to ideas presented by Helfer, Nuyts and others [18].

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A The Lorentz group [13]

The Lorentz group is usually given by its action on the Minkowski vector space $\mathbb{E}_{1,3}$ with the metric $\eta = \text{diag}(1; -1, -1, -1)$. By definition, the Lorentz group $O(1, 3)$ preserves the invariant $x \cdot y = x^\mu \eta_{\mu\nu} y^\nu = x^T \eta y$, $x, y \in \mathbb{E}_{1,3}$, i.e.

$$\Lambda x \cdot \Lambda y = x^T \Lambda^T \eta \Lambda y = x \cdot y \quad \Rightarrow \quad \Lambda^T \eta \Lambda = \eta, \quad (\text{A1})$$

where $O_{1,3} \ni \Lambda : \mathbb{E}_{1,3} \ni x \rightarrow \Lambda x \in \mathbb{E}_{1,3}$. In the matrix form one obtains

$$(\Lambda x)^\mu = \Lambda^\mu{}_\nu x^\nu \quad \text{and} \quad \Lambda^\mu{}_\nu \eta_{\mu\rho} \Lambda^\rho{}_\sigma = \eta_{\nu\sigma}.$$

The group $O_{1,3}$ is topologically homeomorphic to $O_3 \times \mathbb{R}^3$, and the number of connected components is four. Henceforth we will consider mainly the component connected to unity, $\text{Lor}_{1,3} = \text{SO}_{1,3}^0$, called the proper orthochronous Lorentz group,

$$\text{Lor}_{1,3} = \{\Lambda \in \text{M}_3(\mathbb{R}) : \Lambda^T \eta \Lambda = \eta, \det \Lambda = 1, \Lambda^0{}_0 \geq 1\}. \quad (\text{A2})$$

$\text{Lor}_{1,3}$ is a normal subgroup of $O_{1,3}$. The Lorentz group of all proper orthochronous Lorentz transformations of coordinates on the Minkowski space is a six-parameter matrix Lie group. The domain of the six parameters is given by

$$D = \{\eta_1, \eta_2, \eta_3, \omega_1, \omega_2, \omega_3 : \eta_i \in \mathbb{R}, -\pi < \omega_1 \leq \pi, 0 < \omega_2 \leq \pi, -\pi < \omega_3 \leq \pi\},$$

where the boundary points are topologically identified. The resulting region is homeomorphic to $\mathbb{R}^3 \times \mathbb{P}_3$ where \mathbb{P}_3 is the three-dimensional projective space isomorphic to the three-dimensional unit sphere. Therefore, $\text{Lor}_{1,3}$ is locally compact and doubly connected, path connected, simple and reductive. The universal covering group is $\text{SL}_2(\mathbb{C})$, i.e.

$$\text{Lor}_{1,3} = \text{SL}_2(\mathbb{C}) / \mathbb{Z}_2 = \text{SO}_4(\mathbb{C})_{\mathbb{R}} \quad (\text{A3})$$

where the last expression denotes the real form of $\text{SO}_4(\mathbb{C})$.

A.1 Matrix representation

Assuming the Minkowski metric, it is convenient to write the defining representation for Λ blockwise,

$$\Lambda = \begin{pmatrix} A & \vec{B}^T \\ \vec{C} & D \end{pmatrix} \quad (\text{A4})$$

where $A = \Lambda^0_0 \geq 1$, $B_k = \Lambda^0_k$, $C_k = \Lambda^k_0$, and

$$D = (D^i_j) \in \text{GL}(3, \mathbb{R}).$$

For $\Lambda \in \text{Lor}_{1,3}$, one has

$$\Lambda^T \eta \Lambda = \eta \Rightarrow \begin{cases} A^2 = 1 + \vec{C}^T \vec{C} = 1 + |\vec{C}|^2 \\ D^T D = \mathbb{1}_3 + \vec{B} \vec{B}^T \\ A \vec{B} - D^T \vec{C} = 0 \end{cases} \quad (\text{A5.1})$$

$$\Lambda \eta \Lambda^T = \eta \Rightarrow \begin{cases} A^2 = 1 + \vec{B}^T \vec{B} = 1 + |\vec{B}|^2 \\ D D^T = \mathbb{1}_3 + \vec{C} \vec{C}^T \\ A \vec{C} - D \vec{B} = 0 \end{cases} \quad (\text{A5.2})$$

Using these equations, it is easy to see that

$$\Lambda^{-1} = \begin{pmatrix} A & -\vec{C}^T \\ -\vec{B} & D^T \end{pmatrix}.$$

Because of $\det \Lambda = 1$, one has $\det D = A \geq 1$. One can use the equations to rewrite

$$\vec{C} = \frac{1}{A} D \vec{B} \quad \text{or} \quad \vec{B} = \frac{1}{A} D^T \vec{C}$$

to obtain

$$\Lambda = \begin{pmatrix} A & \vec{B}^T \\ \frac{1}{A} D \vec{B} & D \end{pmatrix} = \begin{pmatrix} A & \frac{1}{A} \vec{C}^T D \\ \vec{C} & D \end{pmatrix}. \quad (\text{A6})$$

As a consequence of this, one can apply a polar decomposition [14] to the elements of the Lorentz group $\text{Lor}_{1,3}$, $\Lambda = QP = P'Q$ where Q is orthogonal and P, P' are real symmetric positive definite. Using the ansatz

$$Q = \begin{pmatrix} 1 & 0 \\ 0 & R \end{pmatrix}, \quad P = \begin{pmatrix} A & \vec{B}^T \\ \vec{B} & D_P \end{pmatrix}, \quad P' = \begin{pmatrix} A & \vec{C}^T \\ \vec{C} & D'_P \end{pmatrix} \quad (\text{A7})$$

with $R \in \text{SO}_3$ and D_P, D'_P symmetric, one obtains

$$R = \frac{1}{1+A}(D + AD^{-1T}), \quad D_P = \frac{1}{1+A}(A + D^T D), \quad D'_P = \frac{1}{1+A}(A + DD^T). \quad (\text{A8})$$

According to *Tolhoek's theorem* [9], $P = (\Lambda^T \Lambda)^{1/2}$ and $P' = (\Lambda \Lambda^T)^{1/2}$ describe pure Lorentz transformations or boosts, where

$$A = \frac{1}{\sqrt{1 - v^2/c^2}}, \quad \vec{C} = \frac{1}{c} A \vec{v}.$$

Vice versa, $\Lambda = QP(\vec{B}) = P(R\vec{B})Q$ can be written as

$$\Lambda = \begin{pmatrix} A & \vec{B}^T \\ R\vec{B} & R + \frac{1}{1+A} R\vec{B}\vec{B}^T \end{pmatrix} = \begin{pmatrix} A & \vec{C}^T R \\ \vec{C} & R + \frac{1}{1+A} \vec{C}\vec{C}^T R \end{pmatrix}, \quad (\text{A9})$$

where $A = \Lambda^0_0 \geq 1$, $\vec{B}, \vec{C} \in \mathbb{R}^3$ and $R \in \text{SO}_3$. The *Principal axis theorem* for the group $\text{Lor}_{1,3}$, finally, tells us that every $\Lambda \in \text{Lor}_{1,3}$ has one of the shapes

$$\begin{aligned} S\Lambda_s S^{-1} &= S \begin{pmatrix} \text{lor } t & 0 \\ 0 & \text{rot } \omega \end{pmatrix} S^{-1} \quad \text{or} \\ S\Lambda_u S^{-1} &= S \begin{pmatrix} 1 & 0 \\ 0 & N \end{pmatrix} S^{-1}, \end{aligned} \quad (\text{10})$$

where

$$\begin{aligned} \text{lor } t &= \begin{pmatrix} \cosh t & \sinh t \\ \sinh t & \cosh t \end{pmatrix}, \quad t \in \mathbb{R}, \\ \text{rot } \omega &= \begin{pmatrix} \cos \omega & -\sin \omega \\ \sin \omega & \cos \omega \end{pmatrix} \in \text{SO}_2, \quad -\pi < \omega \leq \pi. \end{aligned} \quad (\text{11})$$

$S \in \text{Lor}_{1,3}$ is a similarity transformation. Since $\text{spec } \Lambda_s = \{e^t, e^{-t}, e^{i\omega}, e^{-i\omega}\}$, Λ_s is semisimple. On the contrary Λ_u is unipotent. Therefore, the Jordan form of N is

$$\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

and $\text{spec } \Lambda_u = \{1\}$. Finally, we note that for all $\Lambda \in \text{Lor}_{1,3}$, Λ satisfies the minimal equation [15]

$$\Lambda^4 - (\text{tr } \Lambda)\Lambda^3 + \frac{1}{2}((\text{tr } \Lambda)^2 - \text{tr } \Lambda^2)\Lambda^2 - (\text{tr } \Lambda)\Lambda + \mathbb{1}_4 = 0.$$

Though it is a pure algebraic reason, one concludes from polar decomposition that the maximal compact subgroup of $\text{Lor}_{1,3}$ is isomorphic to SO_3 . Indeed, $\text{Lor}_{1,3}$ is isomorphic to $\text{SO}_3(\mathbb{C})$, and SO_3 is the compact real form of the latter. Moreover, since $\text{Lor}_{1,3}$ is stable under the transposition (i.e. $\Lambda \in \text{Lor}_{1,3} \Rightarrow \Lambda^T \in \text{Lor}_{1,3}$), $\text{Lor}_{1,3}$ is a linear reductive group, for which the maximal compact subgroup K is determined by the Cartan involution

$$\theta : \text{Lor}_{1,3} \ni \Lambda \rightarrow \theta(\Lambda) = \Lambda^{-1T} \in \text{Lor}_{1,3}$$

as $K = \{\Lambda \in \text{Lor}_{1,3} : \theta(\Lambda) = \Lambda^{-1T} = \Lambda\} = \text{SO}_3$. In this setting, $\text{Lor}_{1,3}/\text{SO}_3$ is a symmetric space.

It is important that the maximal simple compact subgroup $\text{SO}_3 \subset \text{Lor}_{1,3}$ determines the internal symmetry of massive particles, i.e. the spin. On the other hand, the maximal solvable noncompact subgroup called Borel subgroup $\text{Bor}_{1,3} \subset \text{Lor}_{1,3}$ determines the helicity of massless particles. Notice that $\text{Bor}_{1,3}$ is a semidirect product of the abelian subgroups \mathcal{T}_2 and $\text{Tor}_{1,3}$, $\text{Bor}_{1,3} = \mathcal{T}_2 \rtimes \text{Tor}_{1,3}$, where $\text{Tor}_{1,3}$ is the maximal Torus of $\text{Lor}_{1,3}$.

A.2 Generators of $\text{Lor}_{1,3}$

In order to linearise the group $\text{Lor}_{1,3}$, one can simply differentiate it and evaluate the derivative at the identity element of the group. The tangent space at the identity element is the Lie algebra

$$\mathfrak{lor}_{1,3} = \{X \in \mathbb{M}_4(\mathbb{R}) : e^{tX} \in \text{Lor}_{1,3} \forall t \in \mathbb{R}\}. \quad (12)$$

According to *Lie's theorem*, the exponential map $\exp : \mathfrak{lor}_{1,3} \rightarrow \text{Lor}_{1,3}$ is surjective. Therefore, any element $\Lambda \in \text{Lor}_{1,3}$ that is close to the unity can be written as exponential of an element $X \in \mathfrak{lor}_{1,3}$. From Eq. (A1) one concludes that the defining equation for an element $X \in \mathfrak{lor}_{1,3}$ is given by

$$X^T \eta + \eta X = 0. \quad (13)$$

Using the infinitesimal transformation

$$\Lambda^\mu{}_\nu = \eta^\mu{}_\nu + \omega^\mu{}_\nu \equiv \eta^\mu{}_\nu - \frac{1}{2}(\omega_{\rho\sigma} e^{\rho\sigma})^\mu{}_\nu, \quad (14)$$

the defining equation (13) gives $\omega_{\mu\nu} = -\omega_{\nu\mu}$, and one obtains six independent parameters $\omega_{\mu\nu}$ and generators $e^{\mu\nu} = -e^{\nu\mu}$. A generic element $\Lambda \in \text{Lor}_{1,3}$ is written as

$$\Lambda(\omega) = \exp\left(-\frac{1}{2}\omega^{\mu\nu} e_{\mu\nu}\right), \quad e^{\mu\nu} = -\frac{\partial}{\partial\omega^{\mu\nu}}\Lambda(\omega)\Big|_{\omega=0}. \quad (15)$$

The six independent generators $e^{\mu\nu}$ have the form

$$(e^{\mu\nu})^\rho{}_\sigma = -\eta^{\mu\rho}\eta^\nu{}_\sigma + \eta^{\nu\rho}\eta^\mu{}_\sigma \quad (16)$$

and obey the commutation relation

$$[e^{\mu\nu}, e^{\rho\sigma}] = \eta^{\mu\rho}e^{\nu\sigma} + \eta^{\nu\sigma}e^{\mu\rho} - \eta^{\mu\sigma}e^{\nu\rho} - \eta^{\nu\rho}e^{\mu\sigma}. \quad (17)$$

The defining equation (13) applies to the generators in the form

$$e^{\mu\nu T}\eta + \eta e^{\mu\nu} = 0. \quad (18)$$

The minimal equation for $x \in \text{lor}_{1,3}(\equiv \text{so}_{1,3})$ is given by

$$X^4 - \frac{1}{2}(\text{tr } X^2)X^2 + (\det X)\mathbb{1}_4 = 0,$$

and $\det X \leq 0$, $\text{tr } X = 0$.

A.3 Cartan decomposition [16]

Let $\vec{e}_{(i)}$, $i = 1, 2, 3$ be an orthogonal triad for \mathbb{R}^3 defined by¹

$$(\vec{e}_{(i)}^T)_j = (\vec{e}_{(i)})^j = \delta_i^j, \quad \vec{e}_{(i)}^T \vec{e}_{(j)} = \delta_{ij}, \quad (\vec{e}_{(i)} \times \vec{e}_{(j)})^k = \epsilon_{ij}{}^k$$

($1 = \epsilon^{0123} = -\epsilon_{0123} \equiv -\epsilon_{123}$). In terms of this triad the generators $e^{\mu\nu}$ are given by

$$e_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & f_{ij} \end{pmatrix} \equiv \epsilon_{ijk} D^k$$

¹The notation $(X^T)_\mu \equiv X^\mu$ is used in the following.

or, vice versa,

$$D_i = -\frac{1}{2}\epsilon_{0ijk}e^{jk} = \begin{pmatrix} 0 & 0 \\ 0 & -\epsilon_{ijk}\vec{e}_{(j)}\vec{e}_{(k)}^T \end{pmatrix} = -D^i, \quad (\text{A19.1})$$

where $(D_i)^\mu{}_\nu = \epsilon_{0j}{}^\mu{}_\nu$, $f_{ij} = \vec{e}_{(i)}\vec{e}_{(j)}^T = \vec{e}_{(j)}\vec{e}_{(i)}^T$, and

$$e_{0i} = \begin{pmatrix} 0 & \vec{e}_{(i)}^T \\ \vec{e}_{(i)} & 0 \end{pmatrix} \equiv B_i, \quad (\text{A19.2})$$

where $(B_i)^\mu{}_\nu = -\eta_0{}^\mu\eta_{i\nu} + \eta_{0\nu}\eta_i{}^\mu$. A general element $X \in \text{lor}_{1,3}$ has the form

$$X = \vec{\omega}\vec{D} - \vec{\eta}\vec{B} = \begin{pmatrix} 0 & \eta^1 & \eta^2 & \eta^3 \\ \eta^1 & 0 & -\omega^3 & \omega^2 \\ \eta^2 & \omega^3 & 0 & -\omega^1 \\ \eta^3 & -\omega^2 & \omega^1 & 0 \end{pmatrix}. \quad (\text{A20})$$

The corresponding finite Lorentz transformation is given by

$$\Lambda = \exp\left(-\frac{1}{2}\omega^{\mu\nu}e_{\mu\nu}\right) = \exp(\vec{\omega}\vec{D} - \vec{\eta}\vec{B}), \quad (\text{A21})$$

where $\omega^i = \frac{1}{2}\epsilon^{ijk}\omega_{jk}$, $\eta^i = \omega_{0i} = -\eta_i$. The commutation relations can be expressed as

$$\begin{aligned} [D_i, D_j] &= \epsilon_{ijk}D_k, \\ [D_i, B_j] &= \epsilon_{ijk}B_k, \\ [B_i, B_j] &= -\epsilon_{ijk}D_k. \end{aligned} \quad (\text{A22})$$

The compact generators D_i are antisymmetric ($D_i^T = -D_i$) while the noncompact generators B_i are symmetric ($B_i^T = B_i$). As a consequence, the Lorentz algebra $\text{lor}_{1,3}$ (if considered as vector space) is a symmetric Lie algebra with symmetric decomposition

$$\vec{\text{lor}}_{1,3} = \vec{\text{so}}_3 \oplus \vec{\mathfrak{p}}, \quad (\text{A23})$$

where $\vec{\mathfrak{p}} = \text{span}_{\mathbb{R}}\{B_i\}_1^3$. Indeed, so_3 is a subalgebra, $[\text{so}_3, \text{so}_3] = \text{so}_3$, but $[\text{so}_3, \mathfrak{p}] \subset \mathfrak{p}$ and $[\mathfrak{p}, \mathfrak{p}] \subset \text{so}_3$. Given the structure (A20), the generic element $X \in \text{lor}_{1,3}$ can be split up into two parts,

$$X = \begin{pmatrix} 0 & \vec{X}^T \\ \vec{X} & X_{(3)} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & X_{(3)} \end{pmatrix} + \begin{pmatrix} 0 & \vec{X}^T \\ \vec{X} & 0 \end{pmatrix} \quad (\text{A24})$$

where the first part is compact, $X_{(3)}^T = -X^{(3)}$, and contained in \mathfrak{so}_3 , the second part is noncompact and contained in \mathfrak{p} . Notice that \mathfrak{so}_3 and \mathfrak{p} are orthogonal with respect to the Killing form,

$$(\mathfrak{so}_3, \mathfrak{p}) = 0. \quad (\text{A25})$$

Since $\mathfrak{lor}_{1,3}$ is simple, the Cartan–Killing form is nonsingular on \mathfrak{so}_3 and \mathfrak{p} . The symmetric decomposition is determined by the Cartan involution

$$\theta : \mathfrak{lor}_{1,3} \ni X \rightarrow \theta(X) = -X^T = \eta X \eta \in \mathfrak{lor}_{1,3} \quad (\text{A26})$$

which is the only external involutive automorphism for $\mathfrak{lor}_{1,3}$. Indeed, one obtains $\theta(D_i) = -D_i^T = D_i$ and, therefore,

$$\mathfrak{so}_3 = \{X \in \mathfrak{lor}_{1,3} : \theta(x) = +X\} = \text{span}_{\mathbb{R}}\{D_i\}_1^3 \quad (\text{A27.1})$$

is the maximal compact subalgebra of $\mathfrak{lor}_{1,3}$. Similarly,

$$\mathfrak{p} = \{X \in \mathfrak{lor}_{1,3} : \theta(X) = -X\} = \text{span}_{\mathbb{R}}\{B_i\}_1^3 \quad (\text{A27.2})$$

consists of the noncompact elements of $\mathfrak{lor}_{1,3}$. Therefore, one ends up with the cartan decomposition

$$\mathfrak{lor}_{1,3} = \mathfrak{so}_3 + \mathfrak{p}. \quad (\text{A28})$$

The map $\text{SO}_3 \times \mathfrak{P} \rightarrow \text{Lor}_{1,3}$ given by

$$\text{SO}_3 \times \mathfrak{P} \ni (R, X) \rightarrow R \exp X \in \text{Lor}_{1,3}$$

is a diffeomorphism onto $\text{Lor}_{1,3}$. Therefore, the Cartan decomposition (A28) on the level of the Lie algebra $\mathfrak{lor}_{1,3}$ induces the polar decomposition (A7) on the level of the Lie group $\text{Lor}_{1,3}$. The exponential map is a diffeomorphism from the vector space $\vec{\mathfrak{p}}$ of symmetric matrices to the set of positive definite matrices,

$$\exp : \vec{\mathfrak{p}} \ni X = X^T \rightarrow \exp X = (\exp X)^T.$$

A.4 Weyl's unitary trick [16]

It is an algebraic fact that the symmetric spaces appear in pairs. If the Cartan involution induces $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$, it's companion is

$$\mathfrak{g}^{(W)} = \mathfrak{k} + i\mathfrak{p} \equiv \mathfrak{g}_{\mathbb{C}} = \mathbb{C} \otimes \mathfrak{g}. \quad (\text{A29})$$

If \mathfrak{g} is the Lie algebra of a noncompact connected semisimple Lie group G , $\mathfrak{g}^{(W)}$ is the Lie algebra of a second Lie group $G^{(W)}$ which is compact. In this way the noncompact algebras appearing in the Cartan decomposition can be analytically continued to compact algebras by analytic extension,

$$\mathfrak{lor}_{1,3} = \mathfrak{so}_3 + \mathfrak{p} \rightarrow \mathfrak{so}_3 + i\mathfrak{p} \equiv \mathfrak{lor}_{1,3}^{(W)}. \quad (\text{A30})$$

This analytical continuation known as Weyl's unitary trick can be accomplished by using the matrix

$$\Gamma = \begin{pmatrix} i & 0 \\ 0 & \mathbb{1}_3 \end{pmatrix}$$

in the way

$$\mathfrak{lor}_{1,3} \ni X \xrightarrow{\Gamma} X^{(W)} = \Gamma X \Gamma$$

(note that Weyl's unitary trick is not a similarity transformation). One obtains

$$X^{(W)} = \Gamma X \Gamma = \begin{pmatrix} 0 & i\vec{X}^T \\ i\vec{X} & X_{(3)} \end{pmatrix} = \Gamma^{-1}(-\eta X)\Gamma. \quad (\text{A31})$$

and for the basis

$$D_i^{(W)} = \Gamma D_i \Gamma = D_i, \quad B_i^{(W)} = \Gamma B_i \Gamma = iB_i. \quad (\text{A32})$$

Accordingly, the commutation relations (A22) change to their compact form

$$\begin{aligned} [D_i^{(W)}, D_j^{(W)}] &= \epsilon_{ijk} D_k^{(W)}, \\ [D_i^{(W)}, B_j^{(W)}] &= \epsilon_{ijk} B_k^{(W)}, \\ [B_i^{(W)}, B_j^{(W)}] &= \epsilon_{ijk} D_k^{(W)}. \end{aligned} \quad (\text{A33})$$

Because of $(\eta X)^T = -\eta X$, one recovers the algebra $\mathfrak{so}_4(\mathbb{R})$. The last algebra in turn is isomorphic to $\mathfrak{su}_2 \boxplus \mathfrak{su}_2$ where \boxplus denotes the Kronecker sum of algebras, i.e. for $a \in \mathfrak{g}$ and $b \in \mathfrak{h}$ one has

$$a \boxplus b \equiv a \otimes \mathbb{1}_{\mathfrak{h}} + \mathbb{1}_{\mathfrak{g}} \otimes b \in \mathfrak{g} \boxplus \mathfrak{h}. \quad (\text{A34})$$

To conclude, the pair of symmetric algebras $\text{lor}_{1,3}$ and $\text{lor}_{1,3}^{(W)}$ is connected by Weyl's unitary trick,

$$\text{lor}_{1,3} \xrightarrow{\text{Weyl}} \text{lor}_{1,3}^{(W)} = S(\mathfrak{su}_2 \boxplus \mathfrak{su}_2) S^\dagger. \quad (\text{A35})$$

The splitting map

$$S = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 1 \\ -i & 0 & 0 & -i \\ 0 & 1 & 1 & 0 \end{pmatrix} \quad (\text{A36})$$

with $S^\dagger = S^{-1}$ gives the decomposition in the \mathfrak{su}_2 basis,

$$\begin{aligned} D_i^{(W)} &\rightarrow S^\dagger D_i^{(W)} S = m_i \boxplus m_i \Rightarrow D_i = m_i \boxplus m_i, \\ B_i^{(W)} &\rightarrow S^\dagger B_i^{(W)} S = m_i \boxplus (-m_i) \Rightarrow B_i = (-im_i) \boxplus (im_i), \end{aligned} \quad (\text{A37})$$

where $m_j = \frac{i}{2}\sigma_j$, and the decomposition in the \mathfrak{sl}_2 basis,

$$\begin{aligned} D_1 &= \frac{i}{2}(e \boxplus e) + \frac{i}{2}(f \boxplus f), & B_1 &= \frac{1}{2}(e \boxplus (-e)) + \frac{1}{2}(f \boxplus (-f)), \\ D_2 &= \frac{1}{2}(e \boxplus e) - \frac{1}{2}(f \boxplus f), & B_2 &= -\frac{i}{2}(e \boxplus (-e)) + \frac{i}{2}(f \boxplus (-f)), \\ D_3 &= \frac{i}{2}(h \boxplus h), & B_3 &= \frac{1}{2}(h \boxplus (-h)). \end{aligned} \quad (\text{A38})$$

The meaning of Weyl's unitary trick is that the representations of the noncompact group $\text{Lor}_{1,3}$ may be viewed as representations of the compact group $\text{SO}_4 = \text{SU}_2 \times \text{SU}_2 / \mathbb{Z}_2$, where $\mathbb{Z}_2 = \{(\mathbb{1}_2, \mathbb{1}_2), (-\mathbb{1}_2, -\mathbb{1}_2)\}$ is the discrete subgroup. From the unitary nature of the representations of the compact group $\text{SO}(4)$ one conclude the full reducibility of the finite-dimensional representations of $\text{Lor}_{1,3}$. Note that SO_4 is the real compact form of

$\mathrm{SO}_4(\mathbb{C}) = \mathrm{SL}_2(\mathbb{C}) \times \mathrm{SL}_2(\mathbb{C})/\mathbb{Z}_2$, and $\mathrm{Lor}_{1,3}$ is a real noncompact form. Since the simply connected group $\mathrm{SU}(2) \times \mathrm{SU}(2)$ is a universal covering group of the doubly connected group SO_4 , their Lie algebras are isomorphic. Moreover, since \mathfrak{su}_2 is a compact real form of $\mathfrak{sl}_2(\mathbb{C})$, the construction of the representations of the algebra $\mathfrak{lor}_{1,3}$ may be realised by using the representations of $\mathfrak{sl}_2(\mathbb{C})$. Since $\mathrm{SL}_2(\mathbb{C})$ as the topological product $\mathbb{R}^3 \times \mathrm{SU}(2)$ is simply connected, all its representations are single-valued.

A.5 Higher dimensional representations

In the standard basis

$$h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad (\text{A39})$$

of $\mathfrak{sl}_2(\mathbb{C})$ with $[h, e] = 2e$, $[h, f] = -2f$ and $[e, f] = h$, the $(2k+1)$ -dimensional, real representations applied to states $|k, m\rangle$ are given by

$$\begin{aligned} \pi^{(k)}(h)|k, m\rangle &= 2m|k, m\rangle, \\ \pi^{(k)}(e)|k, m\rangle &= \rho_{(m)}^{(k)}|k, m+1\rangle, \\ \pi^{(k)}(f)|k, m\rangle &= \rho_{(-m)}^{(k)}|k, m-1\rangle, \end{aligned} \quad (\text{A40})$$

where $k = 0, \frac{1}{2}, 1, \dots$, $m = -k, -k+1, \dots, k$ and $\rho_{(m)}^{(k)} = \sqrt{(k-m+1)(k+m)}$.

Theorem: Let $2k \in \mathbb{N}$ and let (V, π) be a simple representation of $\mathfrak{sl}_2(\mathbb{C})$ of dimension $2k+1$. Then

1. π is equivalent to $\pi^{(k)}$,
2. the eigenvalues of $\pi^{(k)}(h)/2$ are $\{-k, -k+1, \dots, k\} = \mathrm{spec} \frac{1}{2}\pi^{(k)}(h)$,
3. if $0 \neq v \in V$ verifies $\pi^{(k)}(e)v = 0$, then $\pi^{(k)}(h)v = 2kv$,
i.e. $\pi^{(k)}(h)$ and $\pi^{(k)}(e)$ have the common eigenvector $|k, k\rangle$,

4. if $0 \neq v \in V$ verifies $\pi^{(k)}(f)v = 0$, then $\pi^{(k)}(h)v = -2kv$,

i.e. $\pi^{(k)}(h)$ and $\pi^{(k)}(f)$ have the common eigenvector $|k, -k\rangle$

Since $\mathfrak{su}(2)$ is the compact real form of $\mathfrak{sl}_2(C)$ and the generators of $\mathfrak{su}(2)$ are given by

$$m_1 = \frac{i}{2}(e + f), \quad m_2 = \frac{1}{2}(e - f), \quad m_3 = \frac{i}{2}h, \quad (\text{A41})$$

one can accordingly define irreducible representations of $\mathfrak{su}(2)$ given by

$$\pi^{(k)}(m_1), \quad \pi^{(k)}(m_2) \quad \text{and} \quad \pi^{(k)}(m_3). \quad (\text{A42})$$

Following the procedure given before, the real Lie algebra $\mathfrak{lor}_{1,3}$ may be identified via Weyl's unitary trick with the algebra $\mathfrak{lor}_{1,3}^{(W)}$ which splits into a Kronecker sum of two algebras $\mathfrak{su}(2)$, $\mathfrak{lor}_{1,3}^{(W)} \sim \mathfrak{su}_2 \boxplus \mathfrak{su}_2$. If $\pi^{(k)}$ and $\pi^{(l)}$ are representations of \mathfrak{su}_2 on the vector spaces $V^{(k)}$ and $V^{(l)}$, $\pi^{(k)} \otimes \pi^{(l)}$ is a representation of the Lie algebra $\mathfrak{lor}_{1,3}^{(W)}$ on $V^{(k)} \otimes V^{(l)}$, defined by

$$m_i \boxplus m_j \rightarrow \pi^{(k,l)}(m_i \boxplus m_j) = \pi^{(k)}(m_i) \boxplus \pi^{(l)}(m_j). \quad (\text{A43})$$

The representation $\pi^{(k,l)}$ of the Kronecker sum $\mathfrak{su}_2 \boxplus \mathfrak{su}_2$ on the tensor product basis

$$\{|k, l; m_k, m_l\rangle \equiv |k, m_k\rangle \otimes |l, m_l\rangle, -k \leq m_k \leq k, -l \leq m_l \leq l\}$$

is given by

$$\begin{aligned} \pi^{(k,l)}(m_i \boxplus m_j) |k, l; m_k, m_l\rangle &= \\ &= \sum_{m=-k}^k (\pi^{(k)}(m_i))_{mm_k} |k, l; m, m_l\rangle + \sum_{m=-l}^l (\pi^{(l)}(m_j))_{mm_l} |k, l; m_k, m\rangle. \end{aligned} \quad (\text{A44})$$

Using the representations of $D_i, B_i \in \mathfrak{lor}_{1,3}$ in the \mathfrak{su}_2 basis, one obtains

$$\begin{aligned} \pi^{(k,l)}(D_i) &= \pi^{(k)}(m_i) \boxplus \pi^{(l)}(m_i), \\ \pi^{(k,l)}(B_i) &= (-i\pi^{(k)}(m_i)) \boxplus (i\pi^{(l)}(m_i)) \end{aligned} \quad (\text{A45})$$

with $\pi^{(k)}(m_i)$ given by Eq. (A42).

A.6 Splitting algebra

Theorem: Any finite-dimesional representation of $\mathfrak{lor}_{1,3}$ is isomorphic to $\pi^{(k,l)}$ for some $k, l = 0, \frac{1}{2}, 1, \dots$ and is non-antihermitean. The corresponding representation of $\mathbf{Lor}_{1,3}$ is non-unitary.

The isomorphism between \mathfrak{so}_4 and $\mathfrak{so}_3 \oplus \mathfrak{so}_3$ is easily realised by the choice of the basis

$$\begin{aligned} J_i^{(\varepsilon)} &= \frac{1}{2}(D_i + i\varepsilon B_i) = \frac{1+\varepsilon}{2}m_i \boxplus \frac{1-\varepsilon}{2}m_i, \\ K_i^{(\varepsilon)} &= \frac{1}{2}(D_i - i\varepsilon B_i) = \frac{1-\varepsilon}{2}m_i \boxplus \frac{1+\varepsilon}{2}m_i \end{aligned} \quad (\text{A46})$$

with $J_i^{(\varepsilon)\dagger} = -J_i^{(\varepsilon)}$, $K_i^{(\varepsilon)\dagger} = -K_i^{(\varepsilon)}$ ($\varepsilon = \pm 1$). The commutator relations are given by

$$\begin{aligned} [J_i^{(\varepsilon)}, J_j^{(\varepsilon)}] &= \epsilon_{ijk} J_k^{(\varepsilon)}, \\ [J_i^{(\varepsilon)}, K_j^{(\varepsilon)}] &= 0, \\ [K_i^{(\varepsilon)}, K_j^{(\varepsilon)}] &= \epsilon_{ijk} K_j^{(\varepsilon)}. \end{aligned} \quad (\text{A47})$$

Note that the fact that the Lorentz algebra $\mathfrak{lor}_{1,3}$ can be written as a Kronecker sum $\mathfrak{su}(2) \boxplus \mathfrak{su}(2)$ of two algebras does not mean that $\mathfrak{lor}_{1,3}$ is the same as $\mathfrak{su}(2) \oplus \mathfrak{su}(2)$ or $\mathfrak{lor}_{1,3}^{(W)}$. Rather, they are the anti-hermitean complex representations of $\mathfrak{lor}_{1,3}$.

A.7 Spinor representations [17]

There are two fundamental spinor representations, from which all other may be obtained by tensor product reduction. The Lorentz covariant description needs two sets of relativistic Pauli matrices,

$$(\sigma_\mu) = (\mathbb{1}_2, \vec{\sigma}) \quad \text{and} \quad (\tilde{\sigma}_\mu) = (\mathbb{1}_2, -\vec{\sigma}).$$

The relation between the real Minkowski space $\mathbb{E}_{1,3}$ and the set of all complex hermitean 2×2 matrices \mathbb{H}_2 is given by

$$\mathbb{E}_{1,3} \ni p \rightarrow \sigma(p) = \sigma_\mu p^\mu \in \mathbb{H}_2. \quad (\text{A48})$$

The correspondence is a linear isomorphism,

$$\det \sigma(p) = p^2 = p^\mu p_\mu, \quad (\text{A49})$$

and the characteristic polynomial

$$\det(\sigma(p) - \lambda) = \begin{cases} (p^0 + |\vec{p}| - \lambda)(p^0 - |\vec{p}| - \lambda) & \text{for } p^2 > 0, \\ \lambda(2p^2 - \lambda) & \text{for } p^2 = 0. \end{cases} \quad (\text{A50})$$

Therefore, if $p^2 > 0$, $\sigma(p)$ is positive semidefinite.

Theorem:

1. Let $\sigma(p) \in \mathbb{M}_2(\mathbb{C})$ be positive definite.

If $A \in \mathbb{M}_n(\mathbb{C})$ and $\det A \neq 0$, then $A\sigma(p)A^\dagger$ is positive definite.

2. If $\sigma(p) \in \mathbb{M}_2(\mathbb{C})$ is not positive definite but positive semidefinite and if $A \in \mathbb{M}_n(\mathbb{C})$, then $A\sigma(p)A^\dagger$ is always positive semidefinite and not positive definite.

The fundamental representation $(\frac{1}{2}, 0)$ can be expressed as the commutative diagram

$$\begin{array}{ccc} \text{Lor}_{1,3} \ni \Lambda : & \mathbb{E}_{1,3} \ni p & \longrightarrow \Lambda p \\ & \downarrow \downarrow \sigma & \downarrow \sigma \\ \text{SL}_2(\mathbb{C}) \ni \pm A_\Lambda : & \mathbb{H}_2 \ni \sigma(p) & \longrightarrow A_\Lambda \sigma(p) A_\Lambda^\dagger = \sigma(\Lambda p) \end{array} \quad (\text{A51})$$

The continuous homomorphism relates an element $\Lambda \in \text{Lor}_{1,3}$ to two elements $\pm A_\Lambda \in \text{SL}_2(\mathbb{C})$. The group $\text{SL}_2(\mathbb{C})$ constitutes the universal covering group of $\text{Lor}_{1,3}$, i.e. $\text{Lor}_{1,3} = \text{SL}_2(\mathbb{C})/\mathbb{Z}$ (on the right hand side the realification of $\text{SL}_2(\mathbb{C})$ is understood). Using Pauli's spin matrices one obtains

$$\begin{aligned} A_\Lambda \sigma_\nu A_\Lambda^\dagger &= \sigma_\mu \Lambda^\mu{}_\nu, \\ \Lambda^\mu{}_\nu &= \frac{1}{2} \text{tr}(\sigma_\mu A_\Lambda \sigma_\nu A_\Lambda^\dagger), \\ A_\Lambda &= \frac{\Lambda^{\mu\nu} \sigma_\mu \tilde{\sigma}_\nu}{(\frac{1}{2} \Lambda^{\alpha\beta} \Lambda^{\gamma\delta} \text{tr}(\sigma_\alpha \tilde{\sigma}_\beta \sigma_\delta \tilde{\sigma}_\gamma))^{1/2}}, \end{aligned} \quad (\text{A52})$$

where

$$4(\text{tr } A\Lambda^\dagger)^2 = \frac{1}{2}\Lambda^{\mu\nu}\Lambda^{\rho\sigma}\sigma_\mu\tilde{\sigma}_\nu\sigma_\sigma\tilde{\sigma}_\rho = (\text{tr } \Lambda)^2 - \text{tr } \Lambda^2 + 4 + i\Lambda^{\mu\nu}\Lambda^{\rho\sigma}\epsilon_{\mu\nu\rho\sigma}$$

(note the different order of indices). Suppose that Λ and A_Λ are given by $A_\Lambda\sigma_\nu A_\Lambda^\dagger = \sigma_\mu\Lambda^\mu{}_\nu$, one can write

$$\begin{aligned} A_\Lambda^\dagger\sigma_\nu A_\Lambda &= \sigma_\mu(\Lambda^T)^\mu{}_\nu, \\ A_\Lambda^\dagger\tilde{\sigma}_\nu A_\Lambda &= \tilde{\sigma}_\mu(\Lambda^{-1})^\mu{}_\nu, \\ A_\Lambda\tilde{\sigma}_\nu A_\Lambda^\dagger &= \tilde{\sigma}_\mu(\Lambda^{-1T})^\mu{}_\nu. \end{aligned}$$

Defining $\Lambda = \exp(-\frac{1}{2}\omega^{\mu\nu}e_{\mu\nu})$ and $A = \exp(-\frac{1}{2}\omega^{\mu\nu}e_{\mu\nu})$, for the representation $(\frac{1}{2}, 0)$ one obtains

$$\begin{aligned} m_{\mu\nu} &= \frac{1}{4}(e_{\mu\nu})^{\alpha\beta}\sigma_\alpha\tilde{\sigma}_\beta = -\frac{1}{4}(\sigma_\mu\tilde{\sigma}_\nu - \sigma_\nu\tilde{\sigma}_\mu), \\ m_i &= -\frac{1}{2}\epsilon_{ijk}m^{jk} = \frac{i}{2}\sigma_i, \\ m_{0j} &= \frac{1}{2}\sigma_j = -im_j. \end{aligned} \tag{A53}$$

The second fundamental representation $(0, \frac{1}{2})$ is defined by the commutative diagram

$$\begin{array}{ccc} \text{Lor}_{1,3} \ni \Lambda : & \mathbb{E}_{1,3} \ni p & \longrightarrow \Lambda p \\ & \downarrow \downarrow \tilde{\sigma} & \downarrow \tilde{\sigma} \\ \text{SL}_2(\mathbb{C}) \ni \pm B_\Lambda : & \mathbb{H}_2 \ni \tilde{\sigma}(p) & \longrightarrow B_\Lambda \tilde{\sigma}(p) B_\Lambda^\dagger = \tilde{\sigma}(\Lambda p) \end{array}$$

where $\tilde{\sigma} = C^{-1}\sigma^T C = C^{-1}\sigma^* C$,

$$C = -i\sigma_2 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

and $\tilde{\sigma}(p) = 2p_0 - \sigma(p)$. The properties of the Pauli matrices yield

$$\begin{aligned} B_\Lambda \tilde{\sigma}_\nu B_\Lambda^\dagger &= \tilde{\sigma}_\mu \Lambda^\mu{}_\nu, \\ \Lambda^\mu{}_\nu &= \frac{1}{2} \text{tr}(\tilde{\sigma}_\mu B_\Lambda \tilde{\sigma}_\nu B_\Lambda^\dagger), \\ B_\Lambda &= \frac{\Lambda^{\mu\nu} \tilde{\sigma}_\mu \sigma_\nu}{(\frac{1}{2} \Lambda^{\alpha\beta} \Lambda^{\gamma\delta} \text{tr}(\tilde{\sigma}_\alpha \sigma_\beta \tilde{\sigma}_\delta \sigma_\gamma))^{1/2}} \end{aligned}$$

and

$$4(\text{tr } B_\Lambda^\dagger)^2 = \frac{1}{2} \Lambda^{\mu\nu} \Lambda^{\rho\sigma} \text{tr}(\tilde{\sigma}_\mu \sigma_\nu \tilde{\sigma}_\sigma \sigma_\rho) = (\text{tr } \Lambda)^2 - \text{tr } \Lambda^2 + 4 - i \Lambda^{\mu\nu} \Lambda^{\rho\sigma} \epsilon_{\mu\nu\rho\sigma}.$$

Defining the generators of the representation $(0, \frac{1}{2})$ by $B_\Lambda = \exp(-\frac{1}{2} \omega^{\mu\nu} \tilde{m}_{\mu\nu})$, one obtains

$$\begin{aligned} \tilde{m}_{\mu\nu} &= \frac{1}{4} (e_{\mu\nu})^{\alpha\beta} \tilde{\sigma}_\alpha \sigma_\beta = -\frac{1}{4} (\tilde{\sigma}_\mu \sigma_\nu - \tilde{\sigma}_\nu \sigma_\mu), \\ \tilde{m}_i &= -\frac{1}{2} \epsilon_{ijk} \tilde{m}^{jk} = -\frac{i}{2} \tilde{\sigma}_i = \frac{i}{2} \sigma_i, \\ \tilde{m}_{0j} &= \frac{1}{2} \tilde{\sigma}_j = -\frac{1}{2} \sigma_j = i \tilde{m}_j. \end{aligned}$$

The two nonequivalent fundamental representations $A_\Lambda = A_\Lambda(\frac{1}{2}, 0)$ and $B_\Lambda = A_\Lambda(0, \frac{1}{2})$ are related by

$$A_\Lambda(0, \frac{1}{2}) = C^{-1} A_\Lambda^\dagger(\frac{1}{2}, 0) C = (A_\Lambda(\frac{1}{2}, 0))^{-1\dagger}.$$

References

- [1] H. Poincaré, “Sur la dynamique de l’électron”, Rendiconti del Circolo matematico di Palermo **21** (1906) 129–176 (sent to the editor on July 23rd, 1905)
- [2] E.P. Wigner, “On Unitary Representations of the Inhomogeneous Lorentz Group”, Annals Math. **40** (1939) 149 [Nucl. Phys. Proc. Suppl. **6** (1989) 9];
E.P. Wigner, “Unitary Representations of the inhomogeneous Lorentz Group including Reflections”, Lecture at the Istanbul Summer School of Theoretical Physics (ed. by F. Gürsey), Gordon and Breach, New York and London, 1962, pp. 37–80
- [3] Y. Ohnuki, “Unitary Representations of the Poincaré Group and Relativistic Wave Equations”, World Scientific, Singapore, 1976;
S. Weinberg, “The Quantum Theory of Fields”, Cambridge Univ. Press, 1995
- [4] R. Shaw, “Unitary representations of the inhomogeneous Lorentz group”, Nuovo Cim. **33** (1964) 1074;

- U.H. Niederer and L. O' Raifeartaigh, "Realizations of the unitary representations of the inhomogeneous space-time groups. I+II", *Fortsch. Phys.* **22** (1974) 111;
- C. Fronsdal, "Unitary Irreducible Representations of the Lorentz Group", *Phys. Rev.* **113** (1959) 1367;
- F.R. Halpern and E. Branscomb, "Wigner's Analysis Of The Unitary Representations Of The Poincare Group", Preprint No. UCRL-12359, University of California, 1965;
- G.W. Mackey, "Induced representations of groups and quantum mechanics", Benjamin, New York 1968;
- D.J. Candlin, "Physical operators and the representations of the inhomogeneous Lorentz group", *Nuovo Cim.* **37** (1965) 1396;
- D.L. Pursey, "General theory of covariant particle equations", *Annals Phys.* **32** (1965) 157;
- G. Feldman and P.T. Mathews, "Poincaré invariance, particle fields, and internal symmetry", *Annals Phys.* **40** (1966) 19;
- N. Dragon, "Currents for Arbitrary Helicity", arXiv:1601.07825 [hep-th]
- [5] S. Sternberg, "Group theory and physics", Cambridge Univ. Press, 1994;
- Y.S. Kim and M.E. Noz, "Theory And Applications Of The Poincare Group," D. Reidel Publishing Company, Dordrecht, Netherlands, 1986
- [6] Armand Borel, "Linear Algebraic Groups", Springer, New York, 1991;
- P. Tauvel, R.W.T. Yu, "Lie Algebras and Algebraic Groups", Springer, New York, 2005;
- R. Goodman, N.R. Wallach, "Symmetry, Representations, and Invariants", Springer, New York, 2009
- [7] D. Kwoh, Senior Thesis, Princeton University, Princeton, N.J., 1970 (unpublished), cited on page 457 in: A.S. Wightman, "Relativistic wave equations as singular hyperbolic systems", *Proc. Symp. Pure Math.* **23** (1973) 441

- [8] S. Weinberg, Nucl. Phys. Proc. Suppl. **6** (1989) 67
- [9] J.M. Jauch, C. Frønsdal, R. Hagedorn and H.A. Tolhoek, “The representations of the Lorentz group in quantum mechanics”, J. Math. Phys. **3** (1958) 1116
- [10] D. Finkelstein, “Internal Structure of Spinning Particles”, Phys. Rev. **100** (1955) 924;
 P. Winternitz and I. Frisch, “Invariant expansions of relativistic amplitudes and subgroups of the proper Lorentz group”, Sov. J. Nucl. Phys. **1** (1965) 889;
 J. Patera, P. Winternitz and H. Zassenhaus, “Continuous subgroups of the fundamental groups of physics”, J. Math. Phys. **16** (1975) 1597;
 F.G. Lastaria, “Lie subalgebras of real and complex orthogonal groups in dimension four”, J. Math. Phys. **40** (1999) 449;
 L. Zhang and X. Xue, “The deformation of Poincare subgroups concerning very special relativity”, Sci. China Phys. Mech. Astron. **57** (2014) 859
- [11] Nathan Jacobson, “Lie Algebras”, Dover Publ. Inc., 1979;
 G.G.A. Bäuerle, E.A. de Kerf, “Lie algebras – Finite and Infinite Dimensional Lie Algebras and Applications in Physics”,
 North-Holland, Elsevier Science Publishers B.V., 1990;
 A.O. Barut and R. Raczka, “Theory Of Group Representations And Applications”,
 World Scientific, Singapore, 1986
- [12] Michele Maggiore, “A Modern Introduction to Quantum Field Theory”,
 Oxford Univ. Press, 2005;
 George Sterman, “An Introduction to Quantum Field Theory”,
 Cambridge Univ. Press, 1994;
 F.J. Yndurain, “Relativistic Quantum Mechanics and Introduction to Field Theory”,
 Springer, New York, 1996

- [13] I. Gel'fand, R. Miklos and Z. Shapiro, "Representations of the Rotation and Lorentz Groups and Their Applications" (engl. transl.), Macmillan, New York, 1963;
 G. Lyubarskii, "The Application of Group Theory in Physics" (engl. transl.), Pergamon Press, Oxford, 1960;
 M. Naimark, "Linear Representations of the Lorentz Group" (engl. transl.), Macmillan, New York, 1964;
 R.U. Sexl, H.K. Urbantke, "Relativity, Groups, Particles – Special Relativity and Relativistic Symmetry in Field and Particle Physics", Springer, New York, 2001;
 Wu-Ki Tung, "Group Theory in Physics", World Scientific, Singapore, 1999;
 A. Das, "The Special Theory of Relativity – A Mathematical Exposition", Springer, New York, 1993;
 A. Aste, "Weyl, Majorana and Dirac fields from a unified perspective", arXiv:1605.00146 [hep-th]
- [14] D. Serre, "Matrices – Theory and Applications", Springer, New York, 2002;
 Abraham A. Ungar, "Parametric Realization of the Lorentz Transformation Group in Pseudo-Euclidean Spaces", arXiv:1505.02301 [math-ph]
- [15] Jason Hanson, "Orthogonal decomposition of Lorentz transformations", Gen. Rel. Grav. **45** (2013) 599
- [16] Robert Gilmore, "Lie Groups, Lie Algebras, and Some of Their Applications", Dover Publ. Inc., 2002
- [17] H. Joos, "On the Representation theory of inhomogeneous Lorentz groups as the foundation of quantum mechanical kinematics", Fortsch. Phys. **10** (1962) 65;
 E.M. Corson, "Introduction to Tensors, Spinors, and Relativistic Wave Equations", Amer. Math. Soc., London and Glasgow, 1953;
 A.J. Macfarlane, "On the Restricted Lorentz Group and Groups Homomorphically

- Related to It”, J. Math. Phys. **3** (1962) 1116;
- R. Ticciati, “Quantum Field Theory for Mathematicians”,
Cambridge Univ. Press, 1999
- [18] A. Helfer, M. Hickman, C. Kozameh, C. Lucey and E.T. Newman,
“Yang–Mills Equations And Solvable Groups”, Phys. Rev. **D36** (1987) 1740;
J. Nuyts and T.T. Wu, “Yang–Mills theory for nonsemisimple groups”,
Phys. Rev. **D67** (2003) 025014;
S.C. Anco, “Symmetry properties of conservation laws”, arXiv:1512.01835 [math-ph]